Generalization of Spectral Flatness Measure for Non-Gaussian Linear Processes

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Abstract—We present an information-theoretic measure for the amount of randomness or stochasticity that exists in a signal. This measure is formulated in terms of the rate of growth of multi-information for every new signal sample of the signal that is observed over time. In case of a Gaussian statistics it is shown that this measure is equivalent to the well-known Spectral Flatness Measure that is commonly used in Audio processing. For non-Gaussian linear processes a Generalized Spectral Flatness Measure is developed, which estimates the excessive structure that is present in the signal due to the non-Gaussianity of the innovation process. An estimator for this measure is developed using Negentropy approximation to the non-Gaussian signal and the innovation process statistics. Applications of this new measure are demonstrated for the problem of voiced/unvoiced determination, showing an improved performance.

Index Terms—Information Redundancy, Negentropy, Spectral Flatness Measure, Voiced / Unvoiced Determination, Musical Signals

I. INTRODUCTION

The problem of determining the amount of randomness that is present in a signal is very fundamental to many problems in signal processing. For instance, one might wish to determine the amount of compression gain achievable for a signal in a communication application, determine the presence of a signal rather than noise in detection application or characterize the signal in terms of its inherent “stochasticity” for segmentation or retrieval and many more. A standard method to measure the amount of correlation structure that exists in a signal is by means of a Spectral Flatness Measure (SFM) [1,2,3]. Sometimes called also “tonality coefficient”, it is used to quantify how much tone-like a sound is, as opposed to being noise-like. The meaning of “tonal” in this context is in the sense of the amount of peaks or resonant structure in the power spectrum, as opposed to flat spectrum of a white noise.

In this paper we consider an information-theoretic view of the SFM as the rate of growth of multi-information for every additional sample of the signal, which we call Marginal Information Redundancy, or Multi-Information Rate (MIR). It is shown that this is equivalent asymptotically to the difference between marginal entropy of the signal and the entropy rate of the process. In other words, MIR represents the difference in entropy or uncertainty, or a difference in compression lower bounds, between a memory-less, independent and identically distributed (i.i.d.) sequence of random variables that have same marginal probability distribution equivalent as the signal, and the limiting entropy of the process after taking into account the statistical dependencies between consecutive samples.

It is shown that MIR equals SFM for Gaussian processes, i.e. for signals that can be described as a Gaussian i.i.d. noise passing through a linear filter. Using estimation terminology we say that SFM is a maximum entropy estimate of MIR for the case of second order statistics and Gaussianity of the residual error, also called an innovation process. In such a case SFM measures the structure or lack of randomness in the process only due to the linear dependencies that exist between signal samples, as caused by the filter. It is known also that for continuous random variables, the Gaussian probability distribution function (pdf) has the maximal entropy among all processes with equal variance.

In case when a non-Gaussian process “drives” a linear system, we would like to take into account the additional structure or the decrease in entropy of the innovation. We show that in this case a correction to SFM can be obtained and estimated from the Negentropy approximation to the differential entropy of the innovation process. We call this measure a Generalized Spectral Flatness Measure (GSFM). Testing GSFM on voiced/unvoiced speech signals we show that this results in a more precise detection of the transition points between the random (unvoiced) and structure (voiced) parts of the signal, as compared to standard SFM.

II. MATHEMATICAL PRELIMINARIES

A. Definitions

Given a random variable $x$, with probability density $f(x)$, the entropy of the distribution is defined as [4]

$$H(x) = -\int f(x) \log f(x) dx$$

The entropy rate can be considered as the average amount of bits per sample resulting from compression of an asymptotically long block of signal samples.
For the joint distribution of two variables $x_1, x_2$, the joint entropy is defined as

$$H(x_1, x_2) = -\int f(x_1, x_2) \log f(x_1, x_2) dx_1 dx_2 \quad (2)$$

The average amount of information that the variable $x_1$ carries about $x_2$ is quantified by the mutual information

$$I(x_1, x_2) = H(x_1) + H(x_2) - H(x_1, x_2) \quad (3)$$

Let us denote by $x_i^n = (x_1, x_2, ..., x_n)$ a vector on $n$ samples. Generalization of the mutual information for $n$ variables (called also multi-information) is given by

$$I(x^n) = \sum_{i=1}^{n} H(x_i) - H(x^n) \quad (4)$$

This function measures the average amount of common information contained in $x_i^n$. Using the mutual information we define marginal information redundancy to be the difference between the common information contained in the variables $x_i^n$ and the set $x_i^{n-1}$, i.e. the additional amount of information that is added when one more variable is observed.

$$\rho(x_i^n) = I(x_i^n) - I(x_i^{n-1}) \quad (5)$$

Since in our application we are considering time ordered samples, this redundancy measure corresponds to the rate of growth of the common information as a function of time. It can be shown that the following relation exists between redundancy and entropy

$$\rho(x_i^n) = H(x_i^n) - H(x_i^{n-1}) \quad (6)$$

This shows that redundancy is the difference between the entropy (or uncertainty) of an isolated sample $x_i^n$ and the reduced uncertainty of $x_i^n$ knowing its past. In the case of a stationary process, the first term on the right side of the equation is independent of $n$, while the second term depends only on the length of the "context", i.e. the size of the vector that contains previous samples of the process. In information theoretic terms, this measure equals, asymptotically, the difference between the entropy of the marginal distribution of the process $x$ and its entropy rate, equally for all $n$.

B. Derivation of the Gaussian MIR Estimator

The estimation of MIR is performed by separate estimation of marginal entropy and entropy rate of the signal $x(t)$. Assuming a Gaussian signal, one can show [5] that the marginal entropy of the process is equal to

$$H(x) = \frac{1}{2} \ln(\frac{1}{2\pi \sigma^2})$$

where $S(\omega)$ is the power spectral density of $x(t)$. The entropy rate of Gaussian process, also called Sinai-Kolmogorov Entropy, is given by

$$H_r(x) = \int \ln S(\omega) d\omega + \log \sqrt{2\pi e} \quad (8)$$

MIR is calculated then as the difference

$$\rho = H(x) - H_r(x) \quad (9)$$

Considering a related quantity

$$e^{-2\rho} = \frac{1}{\sqrt{2\pi}} \frac{\exp(\frac{1}{2} \int \ln S(\omega) d\omega)}{\sqrt{\int S(\omega) d\omega}} \quad (10)$$

this expression is known as Spectral Flatness Measure (SFM) [1]. SFM is a well-known and accepted method for evaluation of the “compressibility” of a process. Using this equality we estimate the MIR from the spectral flatness measure as the log (SFM).

C. Properties of SFM for Linear Processes

It can be shown that $0<\text{SFM}<1$, which also follows from the non-negativity of MIR. SFM=0 corresponds to structured or non-random process, i.e. a process where a large difference exists between the marginal entropy and the “true” entropy of the process that is characterized by its entropy rate. SFM=1 corresponds to a random signal in the sense that no extra information can be obtained by looking at longer blocks of signal samples, i.e. having no additional structure when considering these measurements as a “process”.

Moreover, a close relation exists between MIR and likelihood ratio test of white noise vs. Auto Regressive (AR) signal model hypotheses. For Gaussian independent identically distributed (i.i.d.) random variables with variance $\sigma^2$, the minus of asymptotic mean log likelihood approaches the marginal entropy, i.e. $-\frac{1}{n} \log p(x^n_i) \rightarrow H(x)$ assuming a sufficiently long block of signal measurement. This relation can be proved directly from the definitions of entropy and using the i.i.d. property, which in this particular case also equals the entropy rate.

In the case of a Gaussian AR process of a finite order $p$, long correlations are introduced into the output signal due to the regression relations (filtering operation). Even though these correlations are possibly very long, it is still possible to evaluate the conditional probability of the next measurement given its past, using only a limited number of past observations.
Let us denote by $\mathcal{E}$ the innovation process and by $\mathcal{E}_n$ the innovation variable at time $n$, $\mathcal{E}_n = x_n - \sum_{i=1}^{p} a_i x_{n-i}$.

Knowing $p$ past values of the process $x$ and the vector of AR coefficients $\mathbf{a}$, the conditional probability of $x_n$ equals the probability of the corresponding innovation variable $p(x_n | x_{n-1}, \ldots, x_{n-p}, \mathbf{a}) = p(\mathcal{E}_n)$. Since the innovation process is i.i.d., the log-likelihood for a block of $n$ signal samples becomes

$$
\log P(x_1, x_2, \ldots, x_n | AR \text{ Hypothesis}) = \log P(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n) = \sum_{i=1}^{n} \log p(\mathcal{E}_i) \\
\approx nE[\log p(\mathcal{E})] = -nH(\mathcal{E}) \quad (11)
$$

Note that in this derivation we assumed that AR hypothesis includes also the knowledge of the $p$ initial conditions of AR filter. The left part of the equation, for a sufficiently long block of measurements, approaches minus $n$-times the entropy rate $H_r(x)$ giving the relation

$$
H_r(x) = H(\mathcal{E}) \quad (12).
$$

This results in asymptotic expression for likelihood ratio

$$
\log \frac{P(x_1, x_2, \ldots, x_n | i.i.d. \text{ Hypothesis})}{P(x_1, x_2, \ldots, x_n | AR \text{ Hypothesis})} \approx -n(H(x) - H_r(x)) = -n \rho, \quad (13)
$$

where the nominator depends on the variance of the signal $x$ (marginal entropy assuming an i.i.d. process), and the denominator is estimated using the variance of the innovation process $\mathcal{E}$. This example demonstrates the relation between MIR and likelihood ratio test and shows that it can be obtained from estimates of the marginal entropy of the innovation $\mathcal{E}$ and the marginal entropy of signal $x$.

In case that AR model is known, the innovation process can be obtained by applying an inverse filter to the original sequence $x$. In case when the AR model is unknown, the innovation process can be estimated from residual error as linear predictor that is derived based on Minimal Square Error criterion, estimates the innovation, the relations of equation (12) do not necessarily hold anymore. It can be shown that only in case that the prediction error is independent of the predicted sample, these relations remain valid. For instance, in the case of a linear prediction, the prediction error is orthogonal but not necessarily independent of the predicted sample.

In analogous manner to the relations between MIR and SFM that were established in equation (10), we shall generalize the SFM for processes other than Gaussian AR. This will be done using MIR estimates for non-Gaussian processes. The basic idea is to use the marginal entropy of the non-Gaussian innovation as the estimate for the entropy rate of the original process. The distribution of the innovation can be estimated by using various approximations to non-Gaussian pdf.

### III. Generalization of SFM

#### A. MIR evaluation by Negentropy

Negentropy [6] measures the distance of a random variable (r.v.) from Gaussian distribution.

$$
J(x) = H(x_{Gauss}) - H(x) \quad (15)
$$

where $x_{Gauss}$ means Gaussian r.v. with same covariance matrix as $x$, $R_{x_{Gauss}} = R_x$ . We shall denote the entropy of $x_{Gauss}$ by $H_G(x)$. Since in the following we will be estimating negentropy for i.i.d. processes, we will need the variance only, rather than a complete covariance matrix.

In terms of information, this function measures how much information is “left” after taking into account 2nd order statistics. Moreover, it can be shown [4] that Gaussian distribution has the maximal entropy among all distributions of the same variance (cross correlation). This proves that negentropy $\geq 0$.

Writing the multi-information of a vector $x_n$, we get

$$
I(x^n) = \sum_{i=1}^{n} H(x_i) - H(x^n) \\
= J(x^n) - H_G(x^n) + \sum_{i=1}^{n} H(x_i) \\
= J(x^n) - H_G(x^n) + \sum_{i=1}^{n} (H_G(x_i) - J(x_i)) \\
= J(x^n) - \sum_{i=1}^{n} J(x_i) + I_G(x_i^n) \quad (16)
$$
with $I_G(x^n_i) = -H_G(x^n_i) + \sum H_G(x_i)$ denoting Gaussian multi-information. Using the above relations, we write

$$\rho(x^n_i) = I(x^n_i) - I(x^{n-1}) = J(\varepsilon_n) - J(x_n) + \rho_G(x^n_i)$$

(17)

where we used the relation

$$J(x^n_i) - J(x^{n-1}) = \{H_G(x^n_i) - H(x^n_i)\} - \{H_G(x^{n-1}_i) - H(x^{n-1}_i)\}$$

(18)

$$= H_G(x_n | x_{n-1}^i) - H(x_n | x_{n-1}^i)$$

$$= H_G(\varepsilon_n) - H(\varepsilon_n) = J(\varepsilon_n),$$

with $H_G(\varepsilon_n), H(x_n)$ denoting the Gaussian and true marginal entropies respectively, of the innovation/residual signals. We also introduced a new term $\rho_G(x^n_i) = I_G(x^n_i) - I(x^{n-1}_i)$ to denote the MIR of a Gaussian process, which can be calculated from standard SFM using equations (10) and (14).

B. Generalized SFM Definition

From equation (17) one can see that non-Gaussian information appears as a correction factor $\rho_{WNG} = J(\varepsilon_n) - J(x_n)$ to the Gaussian MIR, thus accounting for the non-Gaussian properties of the white residual signal. Writing MIR for the non-Gaussian case as

$$\rho(x^n_i) = \rho_G(x^n_i) + \rho_{WNG}(x_n)$$

(19),

the Generalized SFM (GSFM) is defined as

$$GSFM(x^n_i) = e^{-2\rho(x^n_i)}$$

(20)

To sum up, for the non-Gaussian case, GSFM results in a combination of two factors: 1) part that depends on $\rho_G$ that captures the “structure” due to linear Gaussian part and 2) $\rho_{WNG}$ that contains the excessive structure due to the white non-Gaussian innovation / residual signal.

In the limit $n \to \infty$, GSFM of the process $x$ can be expressed as

$$GSFM(x^n_i) = SFM(x) \cdot e^{-2(J(\varepsilon) - J(x))}$$

(21),

where we discard the time indexes from both the SFM and the two negentropy functions of the original and the innovation signals. GSFM in equation (21) must be understood under the convention that SFM is measured based on spectrum or correlations estimates using asymptotically growing blocks of signal measurements, while the negentropies $J(\varepsilon), J(x)$ are calculated using marginal entropies. The expression in equation (21) can also be given a precise statistical meaning as a function of limit mean values of the relevant statistics for stationary processes and assuming that we use consistent estimators.

C. GSFM Estimation

In order to calculate GSFM, we need to estimate the innovation process and calculate the Negentropy. Several approximations exist for estimation of the Negentropy [7]. The classical method is using higher-order moments

$$J(x) \approx \frac{1}{12} E\{x^3\}^2 + \frac{1}{48} kurt(x)^2$$

(22),

where the kurtosis is defined by

$$kurt(x) = E\{x^4\} - 3E\{x^2\}^2$$

(23).

This estimate was used for the voiced/unvoiced experiments presented in the next section. Our experience seems to indicate that the MIR estimate is rather insensitive to the particular method of Negentropy estimation, probably due to the fact that we are using differences of Negentropies rather than their absolute values.

The complexity of the GSFM estimate is only marginally higher than the original SFM. This estimate amounts to one additional filtering operation, using an inverse filter that was obtained in the SFM estimation step. The calculation of Negentropy requires three summation operations and raising the signal to powers of 2, 3 and 4. One should note, though, that other algorithms are available for SFM estimation that do not require estimation of an AR filter (such as direct estimation using Periodogram methods), and are less computationally demanding.

IV. Examples

A. Upper bound to MIR for a known input pdf

In the case when the input pdf to a finite order AR system is known, we can analytically evaluate $H(x)$ and thus $H_r(x)$. Calculation of MIR requires also the knowledge of the marginal distribution of $x$ for the calculation of $H_r(x)$. The knowledge of this marginal distribution might be difficult to obtain, but we can evaluate an upper bound to MIR by using a Gaussian estimate for the entropy of the marginal distribution of $x$ instead of its true marginal entropy, since it is known that $H_G(x) \geq H(x)$. This gives an upper bound
\[ \rho \leq H_G(x) - H_f(x) \]
\[ = H_G(x) - H_G(\varepsilon) + H_G(\varepsilon) - H(\varepsilon) \]
\[ = -\frac{1}{2} \ln(SFM) + J(\varepsilon) \quad (24) \]

Using analytical expressions of the entropy of different memoryless input sources, we provide in table 1 an upper bound to MIR.

<table>
<thead>
<tr>
<th>Pdf</th>
<th>( H(\varepsilon) )</th>
<th>Upper bound to MIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>( \log_2 \sqrt{12\sigma^2_{\varepsilon}} )</td>
<td>( \frac{1}{2} \log_2 \frac{\pi \varepsilon}{6} - \frac{1}{2} \ln(SFM) )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( \log_2 \sqrt{2\pi \sigma^2_{\varepsilon}} )</td>
<td>( -\frac{1}{2} \ln(SFM) )</td>
</tr>
<tr>
<td>Laplacian</td>
<td>( \log_2 \sqrt{2e\sigma^2_{\varepsilon}} )</td>
<td>( \frac{1}{2} \log_2 \frac{\pi}{e} - \frac{1}{2} \ln(SFM) )</td>
</tr>
<tr>
<td>Gamma</td>
<td>( \log_2 \sqrt{4\pi e^{-1-C}\sigma^2_{\varepsilon}} )</td>
<td>( \frac{1}{2} \log_2 \left(\frac{3}{2} e^C\right) - \ln(SFM) )</td>
</tr>
</tbody>
</table>

where \( C \) is the Euler constant = 0.5772.

Table 1: Upper bound to MIR for various input pdfs.

It should be noted that this bound on the correction factor has values between 0.1 and 0.7, which might be significant compared to the values of \(- \ln(SFM)\) that are approximately of the same order of magnitude (the inverse of SFM is typically 3 and 16 for long term psd’s of speech and video, respectively, [1] p.57).

B. Voicing Determination in Speech

GSFM was applied to the problem of voicing determination in speech signals. For this experiment we have used speech recordings from the Keele speech database [8]. This database contains simultaneously recorded speech and Laryngograph signals. The Laryngograph signal is used as the reference signal describing the true vocal cord activity.

In our application we have used the Burg Maximum Entropy method order 16. This choice of filter order is common to speech processing, while further increasing the order is not desirable since it might capture pitch correlations. Since the system filter is stable minimal phase, it is invertible.

The innovation process \( \varepsilon \) was obtained by filtering \( x \) with the inverse filter.

Figure 1 shows the result of the voicing experiment that was done using the SFM and GSFM measures. The signal is sampled at 20Khz sampling rate. The estimates were done on signal frames of 512 samples, with overlap of 312 samples between the frames. This overlap corresponded to the voicing estimation of the Laryngograph, provided in the database.

One can see that SFM tends to have high values in some cases when a vocal cord activity is present. GSFM provides a more precise indication of the vocal cord activity. It is interesting to note that start points are detected equally well by both methods. This might be due to the fact that the beginning portions of speech phonemes have a strong transient spectral behavior that is detected sufficiently well by the standard SFM. Additionally, both measures indicate that structure exists in some points where no vocal excitation happens. This can occur due to plosive or whistling fricative sounds that may appear in these segments, having a significant structure even without being actually “voiced”.

B. Characterization of Musical Signals

In this experiment we evaluated \( \rho_G \) and \( \rho_{WNG} \) of different musical instruments in the sustained portion of their sound, i.e. the analyzed signals were middle segments of single notes played by different instruments, not containing the transient and the last decaying part of the sound. All instruments were playing the same note (middle C). Such an analysis is common for obtaining descriptors of musical sounds [3] and has also applications in instrumental acoustics [9]. The motivation for this experiment was to investigate the MIR correction factor \( \rho_{WNG} \) as a possible additional feature for instrument description. The spectral envelope of musical signals that roughly corresponds to the resonant properties of the instrument body does not capture all of the characteristics of a musical sound. In figure 2 we
show the results of $\rho_G$ and $\rho_{WNG}$ estimation for five different instruments: Trumpet (Tpt), French Horn (FrH), Flute, Bassoon and the Cello. The results correspond to analysis of sound segments of a length of approximately six pitch periods and filter of order 50. Sensitivity analysis was done, varying the filter order to as low as 10. The results for most instruments are insensitive to variations in the analysis parameters, except for increased variance of $\rho_{WNG}$ estimates, for the case of the Trumpet at lower order filters.

It is interesting to note that $\rho_{WNG}$ for the Trumpet and the French Horn are negative, thus indicating that the original signal is far from Gaussianity, while the estimated innovation signal, resulting from inverse LP filtering operation that “mixes” the original signal, is closer to being Gaussian. Considering separately the negentropy $J(x)$ of these signals shows indeed that Trumpet and French Horn are far from Gaussianity, while the other instruments are not. In terms of compression gain [4], these results suggest that using LP model for prediction of signals such as the Trumpet or French Horn may result in a reduced compression gain than what is predicted by SFM alone.

V. CONCLUSION

In this paper we presented a generalization of the standard spectral flatness measure using an information theoretic formulation of randomness, as a marginal information redundancy. It was shown that the new measure captures additional information about signal statistics and can be applied to improve voicing determination in speech signals, or possibly provide an additional characteristic for analysis of musical sounds. It is interesting to explore additional signal representations, such as signal transform or sub-band coding methods, for signal information redundancy processing. We are currently considering the use of optimal transform methods and other signal representation for improved estimation of information redundancy. The utility of the principle of information redundancy for additional signal processing applications, such as compression or retrieval, will be considered in the future.

REFERENCES
