Music 270a: Complex Exponentials and Spectrum Representation

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Exponentials

- The exponential function is typically used to describe the natural growth or decay of a system's state.
- An exponential function is defined as

$$x(t) = e^{-t/\tau},$$

where $e = 2.7182...$, and τ is the **characteristic time constant**, the time it takes to decay by $1/e$.

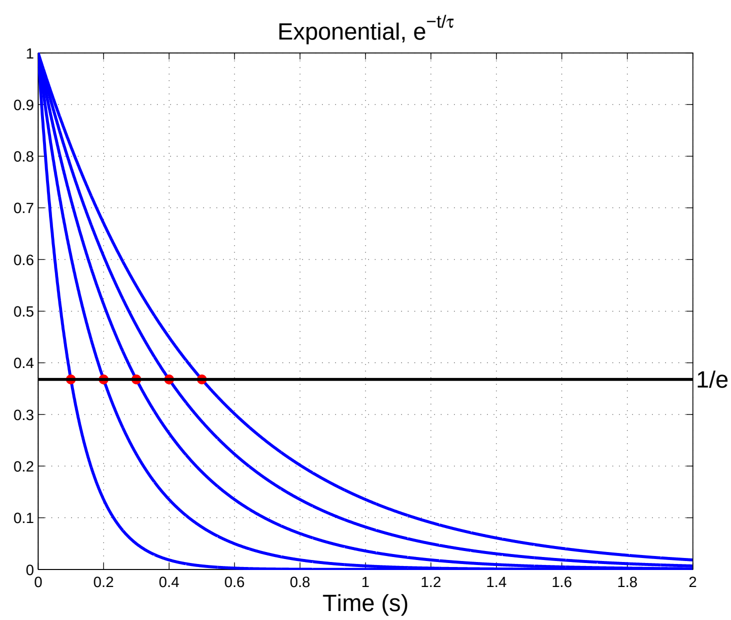


Figure 1: Exponentials with characteristic time constants, .1, .2, .3, .4, and .5

- Both exponential and sinusoidal functions are aspects of a slightly more complicated function.

Complex numbers

- Complex numbers provides a system for
  1. manipulating rotating vectors, and
  2. representing geometric effects of common digital signal processing operations (e.g. *filtering*), in algebraic form.
- In *rectangular* (or *Cartesian*) form, the complex number z is defined by the notation

$$z = x + jy.$$

- The part *without* the j is called the **real** part, and the part with the j is called the **imaginary** part.

Complex Numbers as Vectors

- A complex number can be drawn as a vector, the tip of which is at the point (x, y), where

  $x \triangleq$ the horizontal coordinate—the **real part**,

  $y \triangleq$ the vertical coordinate—the **imaginary part**.

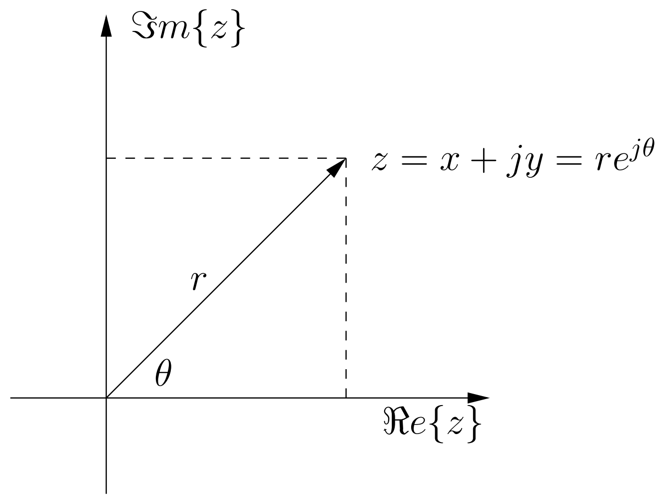


Figure 2: Cartesian and polar representations of complex numbers in the complex plane.

- Thus, the x- and y-axes may be referred to as the **real** and **imaginary** axes, respectively.
- A multiplication by j may be seen as an operation meaning "rotate counterclockwise $90°$ or $\pi/2$ radians".
- Two successive rotations by $\pi/2$ bring us to the negative real axis ($j^2 = -1$), and thus $j = \sqrt{-1}$.

Polar Form

- A complex number may also be represented in **polar form**

  $$z = re^{j\theta},$$

  where the vector is defined by its

  1. length r, and
  2. direction θ (angle with horizontal real x-axis).

- The length of the vector is also called the *magnitude* of z (denoted $|z|$), that is

  $$|z| = r.$$

- The angle with the real axis is called the *argument* of z (denoted $\arg z$), that is

  $$\arg z = \theta.$$

Converting from Cartesian to Polar

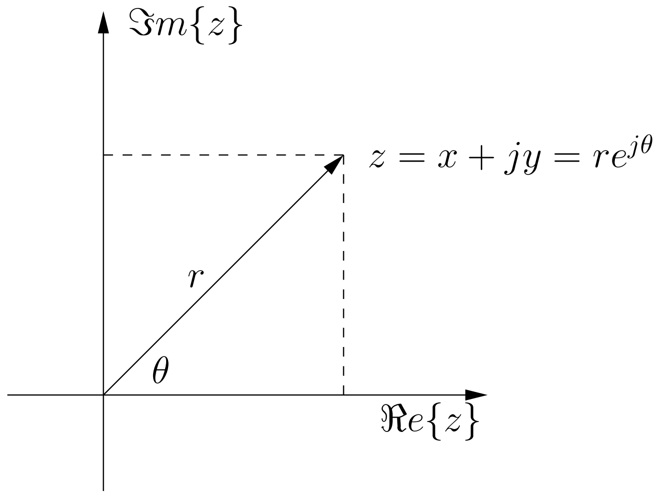


Figure 3: Cartesian and polar representations of complex numbers in the complex plane.

- Using trigonometric identities and the Pythagorean theorem, we can compute:
  1. The Cartesian coordinates(x, y) from the polar variables $r\angle\theta$:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

  2. The polar coordinates from the Cartesian:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Projection and Sinusoidal Motion

- Recall from our previous section on sinusoids that the projection of a rotating sinusoid on the $x-$ and $y-$ axes, traces out a cosine and a sine function respectively.

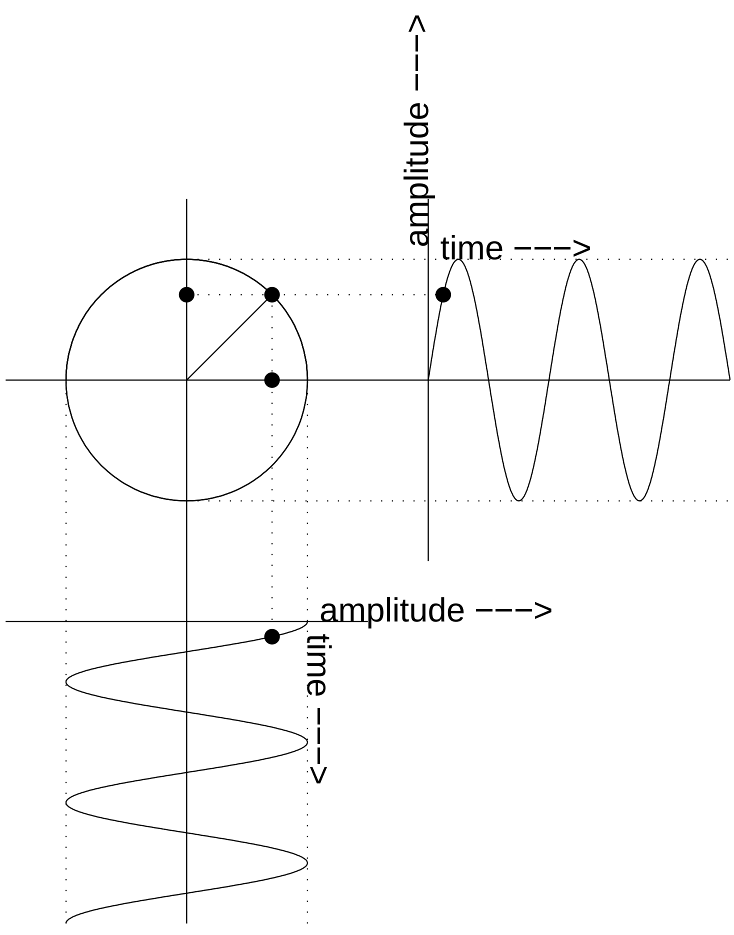


Figure 4: Projection on the $x-$ and $y-$ axis.

Euler's Formula

- From the result of sinusoidal projection, we can see how Euler's famous formula for the complex exponential was obtained:

  $$e^{j\theta} = \cos\theta + j\sin\theta,$$

  valid for any point $(\cos\theta, \sin\theta)$ on a circle of radius one (1).

- Euler's formula can be further generalized to be valid for any complex number z:

  $$z = re^{j\theta} = r\cos\theta + jr\sin\theta.$$

- Though called "complex", these number usually simplify calculations considerably—particularly in the case of multiplication and division.

Complex Exponential Signals

- The complex exponential signal (or *complex sinusoid*) is defined as

$$x(t) = Ae^{j(\omega_0 t + \phi)}.$$

- It may be expressed in Cartesian form using Euler's formula:

$$\begin{aligned} x(t) &= Ae^{j(\omega_0 t + \phi)} \\ &= A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi). \end{aligned}$$

- As with the real sinusoid,
  - A is the *amplitude* given by $|x(t)|$

$$\begin{aligned} |x(t)| &\triangleq \sqrt{\text{re}^2\{x(t)\} + \text{im}^2\{x(t)\}} \\ &\equiv \sqrt{A^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)]} \\ &\equiv A \quad \text{for all } t \\ &\quad (\text{since } \cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) = 1). \end{aligned}$$

  - ϕ is the initial phase
  - ω_0 is the frequency in rad/sec
  - $\omega_0 t + \phi$ is the instantaneous phase, also denoted $\arg x(t)$.

Real and Complex Exponential Signals

How does the Complex Exponential Signal compare to the real sinusoid?

- As seen from Euler's formula, the sinusoid given by $A\cos(\omega_0 t + \phi)$ is the real part of the complex exponential signal. That is,

$$A\cos(\omega_0 t + \phi) = \text{re}\{Ae^{j(\omega_0 t + \phi)}\}.$$

- Recall that sinusoids can be represented by the sum of **in-phase** and **phase-quadrature** components.

$$\begin{aligned}
&A\cos(\omega_0 t + \phi) = \text{re}\{Ae^{j(\omega_0 t + \phi)}\} \\
&= \text{re}\{Ae^{j(\phi + \omega_0 t)}\} \\
&= A\text{re}\{e^{j\phi}e^{j\omega_0 t}\} \\
&= A\text{re}\{(\cos\phi + j\sin\phi)(\cos(\omega_0 t) + j\sin(\omega_0 t))\} \\
&= A\text{re}\{\cos\phi\cos(\omega_0 t) - \sin\phi\sin(\omega_0 t) \\
&\qquad + j(\cos\phi\sin(\omega_0 t) + \sin\phi\cos(\omega_0 t))\} \\
&= A\cos\phi\cos(\omega_0 t) - A\sin\phi\sin(\omega_0 t).
\end{aligned}$$

Inverse Euler Formulas

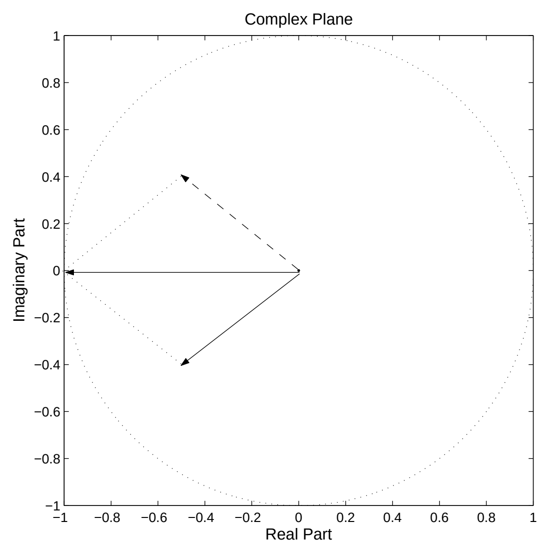
- The **inverse Euler formulas** allow us to write the cosine and sine function in terms of complex exponentials:
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2},$$
and
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$
- This can be shown by adding and subtracting two complex exponentials with the same frequency but opposite in sign,
$$\begin{aligned} e^{j\theta} + e^{-j\theta} &= \cos\theta + j\sin\theta + \cos\theta - j\sin\theta \\ &= 2\cos\theta, \end{aligned}$$
and
$$\begin{aligned} e^{j\theta} - e^{-j\theta} &= \cos\theta + j\sin\theta - \cos\theta + j\sin\theta \\ &= 2j\sin\theta. \end{aligned}$$
- A real cosine signal is actually **composed of two complex exponential signals**:
  1. one with a **positive frequency**
  2. one with a **negative frequency**

Complex Conjugate

- The complex conjugate $\overline{z}$ of a complex number $z = x + jy$ is given by

$$\overline{z} = x - jy.$$

- A real cosine can be represented in the complex plane as the sum of two complex rotating vectors (scaled by 1/2) that are complex conjugates of each other.



- The negative frequencies that arise from the complex exponential representation of the signal, will greatly simplify the task of signal analysis and spectrum representation.

Conjugate Symmetry (Hermitian)

- A complex sinusoid $e^{j\omega t}$ consists of $\mathbf{one}$ frequency ω.
- A real sinusoid $\sin(\omega t)$ consists of $\mathbf{two}$ frequencies ω and $-\omega$.
- Every real signal, therefore, consists of an equal contribution of positive and negative frequency components.
- If $X(\omega)$ denotes the spectrum of the real signal $x(t)$, then $X(\omega)$ is conjugate symmetric (Hermitian), implying

$$|X(-\omega)| = |X(\omega)|$$

  and

$$\angle X(-\omega) = -\angle X(\omega)$$

- It is sometimes easier to use the "less complicated" complex sinusoid when doing signal processing.
- Negative frequencies in a real signal may be "filtered out" to produce an *analytic signal*, a signal which has no negative frequency components.

Analytic Signals

- The real sinusoid $x(t) = A\cos(\omega t + \phi)$ can be converted to an *analytic signal*, by generating a **phase quadrature component**,

$$y(t) = A\sin(\omega t + \phi),$$

to serve as the imaginary part.

1. Consider the positive and negative frequency components of a real sinusoid at frequency ω_0:

$$\begin{aligned} x_+ &\triangleq e^{j\omega_0 t} \\ x_- &\triangleq e^{-j\omega_0 t}. \end{aligned}$$

2. Apply a phase shift of $-\pi/2$ radians to the positive-frequency component,

$$y_+ = e^{-j\pi/2}e^{j\omega_0 t} = -je^{j\omega_0 t}$$

and a phase shift of $\pi/2$ to the negative-frequency component,

$$y_- = e^{j\pi/2}e^{-j\omega_0 t} = je^{-j\omega_0 t}.$$

3. Form a new complex signal by adding them together:

$$\begin{aligned} z_+(t) &\triangleq x_+(t) + jy_+(t) = e^{j\omega_0 t} - j^2 e^{j\omega_0 t} = 2e^{j\omega_0 t} \\ z_-(t) &\triangleq x_-(t) + jy_-(t) = e^{-j\omega_0 t} + j^2 e^{-j\omega_0 t} = 0. \end{aligned}$$

Hilbert Transform Filters

- For more complicated signals (which are the sum of sinusoids), the *Hilbert Transform* may be used to shift each sinusoidal component by a quarter cycle.
- When a real signal $x(t)$ and its Hilbert transform $y(t) = \mathcal{H}_t\{x\}$ are used to form a new complex signal

  $$z(t) = x(t) + jy(t),$$

  the signal $z(t)$ is the (complex) *analytic signal* corresponding to the real signal $x(t)$.
- *Problem*: Given the modulated signal

  $$x(t) = A(t)\cos(\omega t).$$

  How do you obtain $A(t)$ without knowing ω?

  Answer: Use the Hilbert tranform to generate the analytic signal

  $$z(t) \approx A(t)e^{j\omega t},$$

  and then take the absolute value

  $$A(t) = |z(t)|.$$

Complex Amplitude or Phasor

- When two complex numbers are multiplied, their magnitudes multiply and their angles add:
$$r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}.$$
- If the complex number $X = Ae^{j\phi}$ is multiplied by the complex exponential signal $e^{j\omega_0 t}$, we obtain
$$x(t) = Xe^{j\omega_0 t} = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\phi)}.$$
- The complex number X is referred to as the **complex amplitude**, a polar representation of the amplitude and the initial phase of the complex exponential signal.
- The complex amplitude is also called a **phasor** as it can be represented graphically as a vector in the complex plane.

Spectrum Representation

- Recall that summing sinusoids of the same frequency but arbitrary amplitudes and phases produces a new single sinusoid of the same frequency.
- Summing several sinusoids of different frequencies will produce a waveform that is no longer purely sinusoidal.
- The **spectrum of a signal** is a graphical representation of the frequency components it contains and their complex amplitudes.
- Consider a signal that is the sum of N sinusoids of arbitrary amplitudes, phases, AND frequencies:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \phi_k)$$

Spectrum Representation cont.

- Using inverse Euler, this signal may be represented as

$$x(t) = A_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j\omega_k t} + \frac{\overline{X}_k}{2} e^{-j\omega_k t} \right\}.$$

- Every signal therefore, can be expressed as a linear combination of complex sinusoids.
- If a signal is the sum of N sinusoids, the spectrum will be composed of $2N+1$ complex amplitudes and $2N+1$ complex exponentials of a certain frequency.

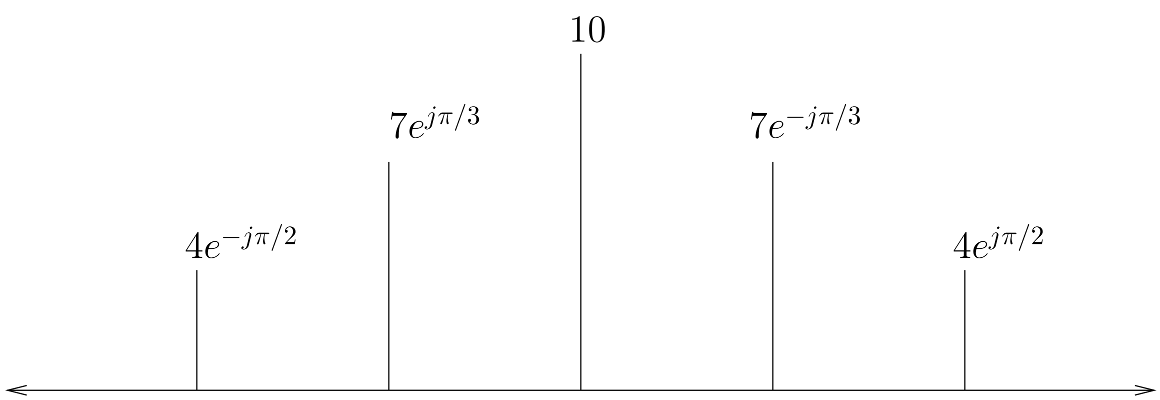


Figure 5: Spectrum of a signal with $N = 2$ components.

Why are phasors important?

- Linear Time Invariant (LTI) systems perform only four (4) operations on a signal: copying, scaling, delaying, adding.
- The output of an LTI system therefore is always a *linear combination* of delayed copies of the input signal(s).
- In a discrete time system, any linear combination of delayed copies of a complex sinusoid may be expressed as

$$y(n) = \sum_{i=1}^{N} g_i x(n - d_i)$$

where g_i is the i^{th} weighting factor, d_i is the i^{th} delay, and

$$x(n) = e^{j\omega nT}.$$

Linear Time Invariant Systems

- Notice, the "carrier term" $x(n) = e^{j\omega nT}$ can be factored out to obtain

$$\begin{aligned} y(n) &= \sum_{i=1}^{N} g_i x(n - d_i) \\ &= \sum_{i=1}^{N} g_i e^{j[\omega(n-d_i)T]} \\ &= \sum_{i=1}^{N} g_i e^{j\omega nT} e^{-j\omega d_i T} \\ &= x(n) \sum_{i=1}^{N} g_i e^{-j\omega d_i T}, \end{aligned}$$

  showing an LTI system can be reduced to a calculation involving only the sum phasors.

- Since every digital signal can be expressed as a linear combination of complex sinusoids, this analysis can be applied to *any* digital signal.

Signals as Vectors

- For the Discrete Fourier Transform (DFT), all signals and spectra are length N:
  - signal $x(n)$ may be real or complex, where $n = 0, 1, ...N - 1$.
- We may regard x as a vector $\underline{x}$ in an N dimensional *vector space*. That is, each sample $x(n)$ is regarded as a *coordinate* in that space.
- Mathematically therefore, a vector $\underline{x}$ is a single point in N-space, represented by a list of coordinates $(x(0), x(1), x(2), ..., x(N - 1)$.

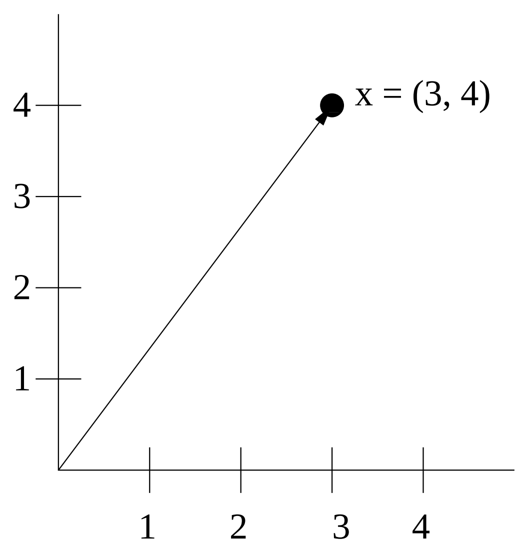


Figure 6: A length 2 signal plotted in 2D space.

Projection, Inner Product and the DFT

- The *coefficient of projection* of a signal x onto another signal y:
  - "a measure of how much y is present in x"
  - is computed using the inner product $\langle x, y \rangle$:

$$\langle x, y \rangle \triangleq \sum_{n=0}^{N-1} x(n)\overline{y(n)}.$$

- The vectors (signals) x and y are said to be *orthogonal* if $\langle x, y \rangle = 0$:

$$x \perp y \Leftrightarrow \langle x, y \rangle = 0$$

- Consider the projection of

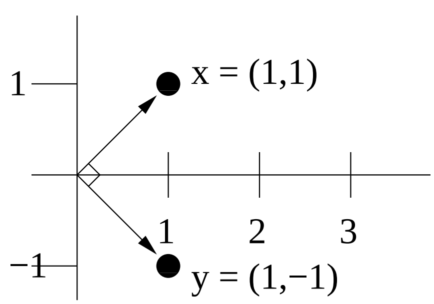


Figure 7: Two orthogonal vectors for $N = 2$

$$\langle x, y \rangle = 1 \cdot \overline{1} + 1 \cdot \overline{(-1)} = 0.$$

Orthogonality of Sinusoids

- Sinusoids are orthogonal at different frequencies if their durations are infinite.
- For length N sampled sinusoidal segments, orthogonality holds for the *harmonics of the sampling rate divided by N*, that is for frequencies

$$f_k = k\frac{f_s}{N}, k = 0, 1, 2, 3, ..., N-1.$$

- These are the only frequencies that have a whole number of periods in N samples.
- The complex sinusoids corresponding to the frequencies f_k are

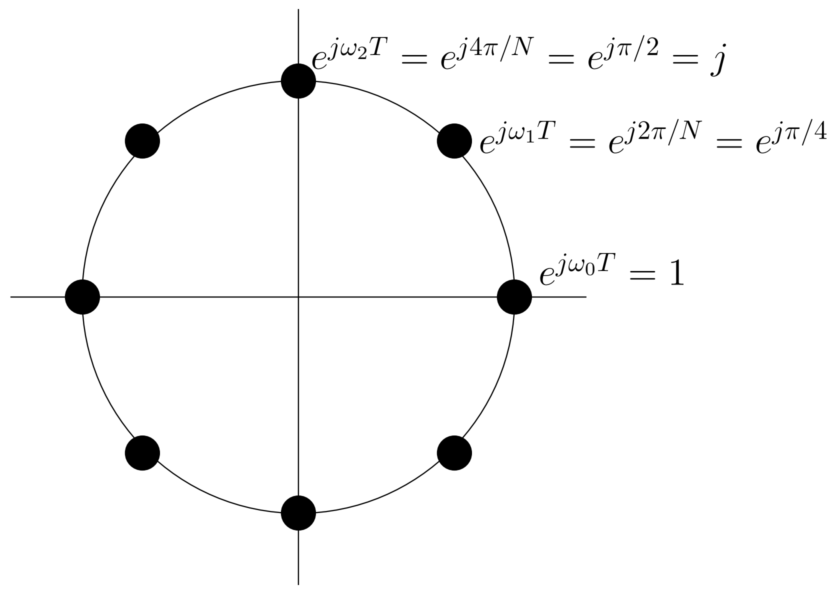
$$s_k(n) \triangleq e^{j\omega_k nT}, \omega_k \triangleq k\frac{2\pi}{N}f_s, k = 0, 1, 2, ..., N-1.$$

These sinusoids are generated by the Nth *roots of unity* in the complex plane, so called since

$$[e^{j\omega_k T}]^N = [e^{jk2\pi/N}]^N = e^{jk2\pi} = 1.$$

DFT Sinusoids

- The Nth roots of unity are plotted below for $N = 8$.



- The sampled sinusoids corresponding to the N roots of unity are given by $(e^{j\omega_k T})^n = e^{j2\pi kn/N}$, and are used by the DFT.
- Taking successively higher integer powers of the root $e^{j\omega_k T}$ on the unit circle, generates samples of the kth DFT sinusoid.
- Since each sinusoid is of a different frequency and each is a harmonic of the sampling rate divided by N, the DFT sinusoids are orthogonal.

DFT

- Recall, one signal $y(\cdot)$ is *projected* onto another signal $x(\cdot)$ using an *inner product* defined by

$$\langle y, x \rangle \triangleq \sum_{n=0}^{N-1} y(n)\overline{x(n)}$$

- If $x(n)$ is a sampled, unit amplitude, zero-phase, complex sinusoid,

$$x(n) = e^{j\omega_k nT}, n = 0, 1, \ldots, N-1,$$

then the inner product computes the *Discrete Fourier Transform* (DFT).

$$\begin{aligned} \langle y, x \rangle &\triangleq \sum_{n=0}^{N-1} y(n)\overline{x(n)} \\ &= \sum_{n=0}^{N-1} y(n)e^{-j\omega_k nT} \\ &\triangleq \text{DFT}_k(y) \triangleq Y(\omega_k) \end{aligned}$$

- $Y(\omega_k)$, the DFT at frequency ω_k, is a measure of the amplitude and phase of the complex sinusoid which is present in the input signal x at that frequency.

Final DFT and IDFT

- The DFT is most often written

$$X(\omega_k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, k = 0, 1, 2..., N-1.$$

- The IDFT is normally written

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(\omega_k)e^{j\frac{2\pi kn}{N}}.$$

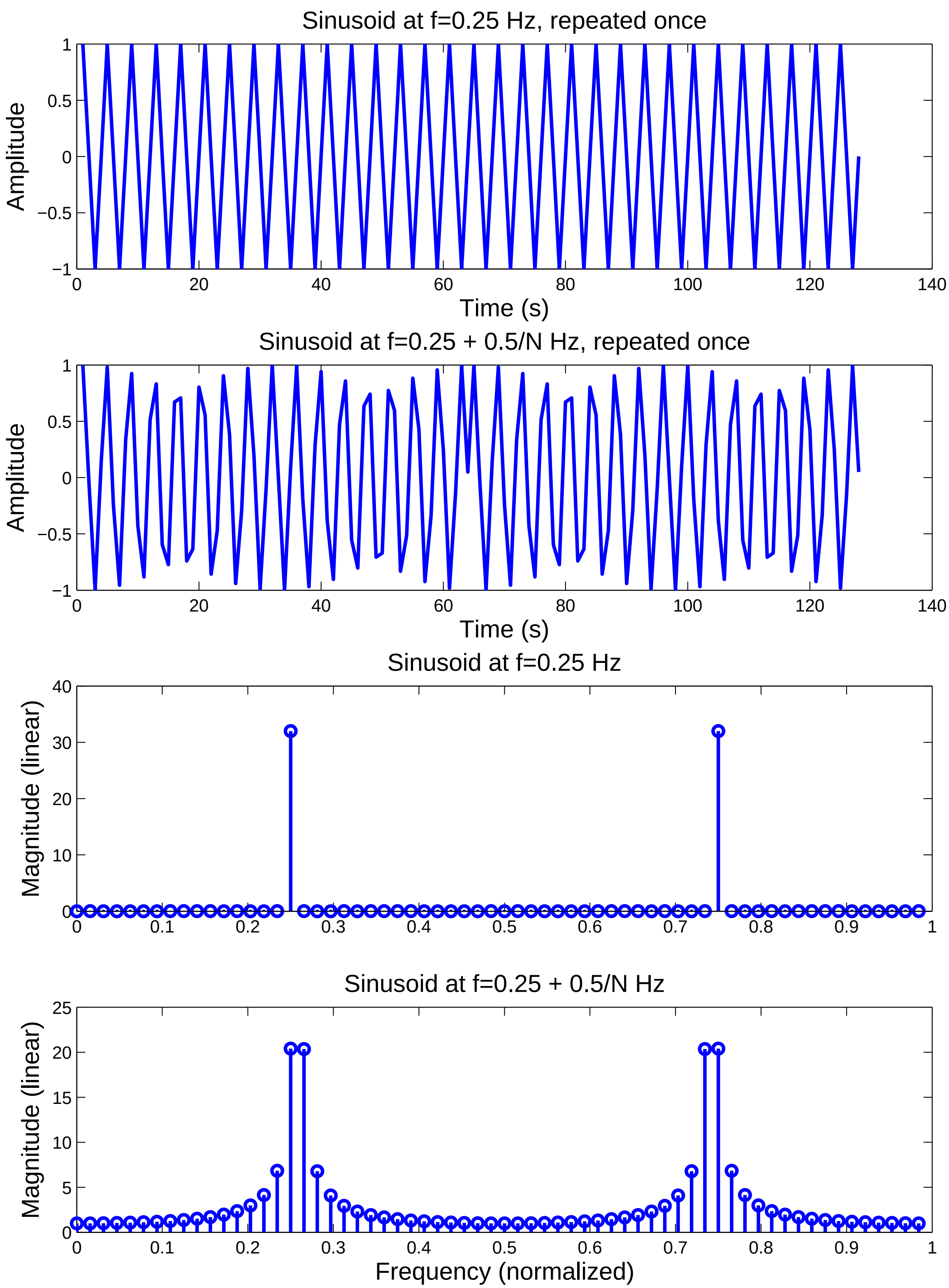
Between the DFT Bins

- Recall that DFT sinusoids are integer multiples of the sampling rate divided by N

$$f_k = k\frac{f_s}{N}, k = 0, 1, 2, ..., N - 1.$$

- The DFT sinusoids are the only frequecies that have a whole number of periods in N samples.
- Consider the periodic extension of a sinusoid lying between DFT bins (see Matlab script betweenBins.m).

- Notice the "glitch" in the middle where the signal beings its forced repetition. This results in spectral "artifacts".


Sinusoid at f=0.25 Hz, repeated once
Amplitude
Time (s)
Sinusoid at f=0.25 + 0.5/N Hz, repeated once
Amplitude
Time (s)
Sinusoid at f=0.25 Hz
Magnitude (linear)
Sinusoid at f=0.25 + 0.5/N Hz
Magnitude (linear)
Frequency (normalized)

Zero-padding

- This problem can be handled, to some extent, by increasing the resolution of the DFT—increasing N by appending zeros to the input signal.

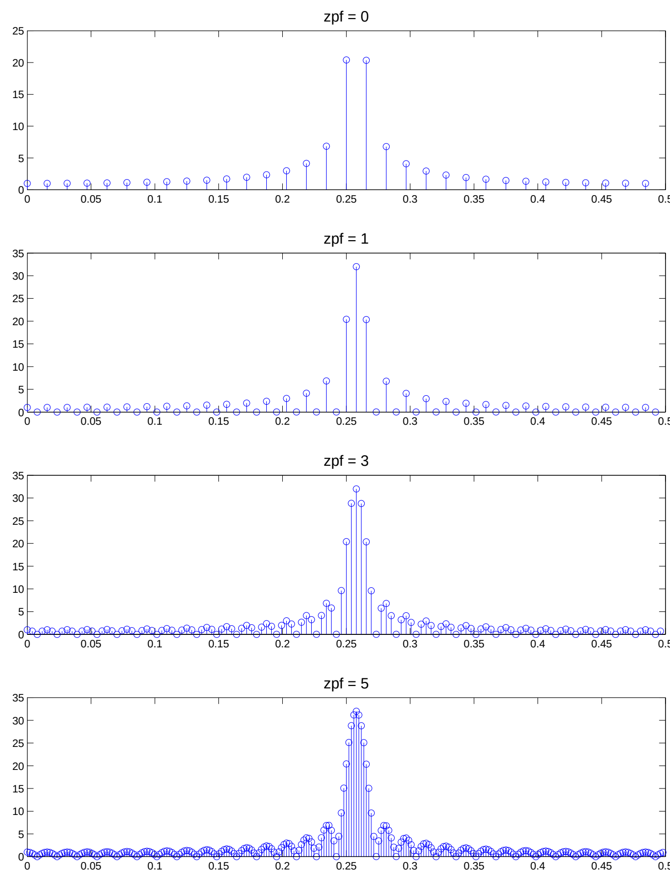


Figure 9: Spectral effect of zero padding.

Windowing

- To further improve the output of the DFT, it is desirable to apply a window, to reduce the effects of the "glitch".
- Applying no window at all is akin to applying a *rectangle* window—selecting a finite segment of length N from a sampled sinusoid.
- The spectral characteristics of a rectangle window can be seen by taking it's spectrum (see windowSpec.m).

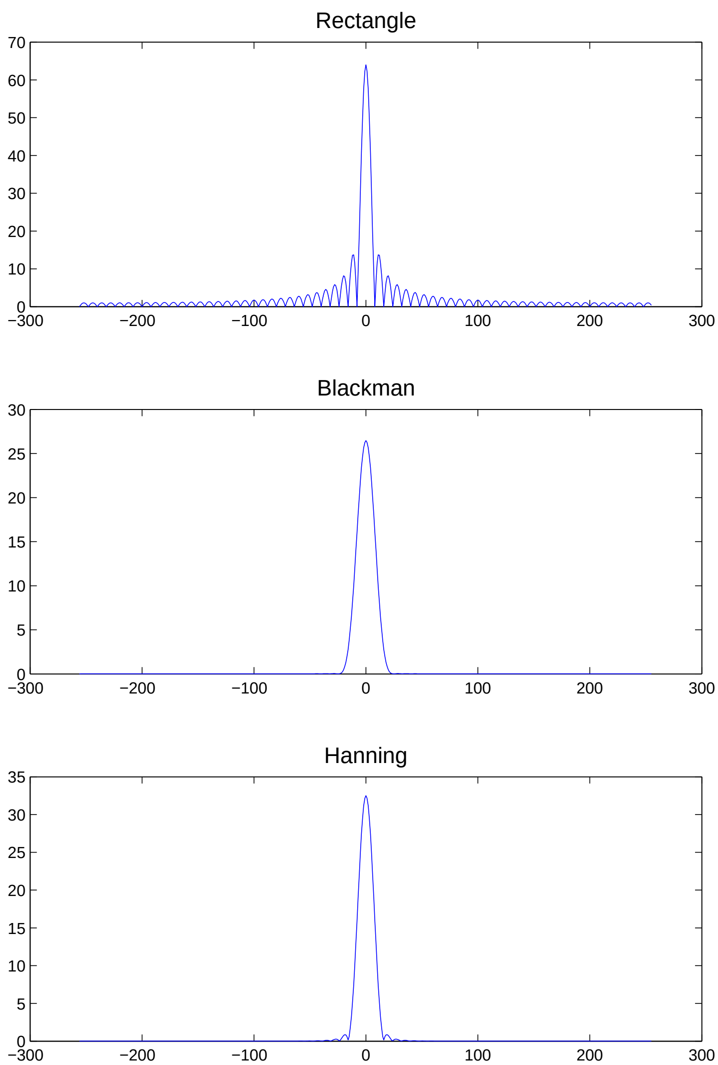


Figure 10: Window Spectra.