Analog Signals

- A **signal**, of which a sinusoid is only one example, is a set, or sequence of numbers.
- The term “analog” refers to the fact that it is “analogous” of the signal it represents.
- A “real-world” signal is captured using a microphone which has a diaphragm that is pushed back and forth according to the compression and rarefaction of the sounding pressure waveform.

![Image of piano, microphone, and electrical signal](image)

Figure 1: The electrical signal used to represent the pressure variations of a sound wave is an analog signal.

- The microphone transforms this displacement into a time-varying voltage—an analog electrical signal.
Continuous-Time Signals

- Analog signals are *continuous* in time.

- A **continuous-time** signal is an *infinite* and *uncountable* sequence of numbers, as are the possible values each number can have.
  
  – that is, between a start and end time, there are infinite possible values for time $t$ and instantaneous amplitude $x(t)$.

- A continuous signal cannot be stored, or processed, in a computer since it would require an infinite amount of data.

- Analog signals must therefore be *discretized*, or *digitized*, to produce a finite set of numbers, for computer use.
Discrete-Time, Digital Signals

- When analog signals are brought into a computer, they must be made *discrete*).
- In a **discrete-time signal**, the number of elements in the set, as well as the possible values of each element, is finite, countable, and can be represented with computer bits, and stored on a digital storage medium.

![Discrete-Time Signal Diagram]

- **Sampling**: the process of taking individual values of a continuous-time signal (at regular time intervals).
Analog to Digital Conversion

- The process by which an analog signal is digitized is called analog-to-digital or “a-to-d” conversion

- analog-to-digital converter (ADC): the device that digitizes (discretizes) an analog signal.

- The ADC must accomplish two (2) tasks:
  1. **sampling**:  
     - take values at regular time intervals;
  2. **quantization**:  
     - assign a number to the value (using limited computer bits).
Sampling

- **Sampling**: process of taking sample values of the continuous waveform at regularly spaced time intervals.

\[
x(t) \xrightarrow{\text{ADC}} x[n] = x(nT_s)
\]

\[T_s = \frac{1}{f_s}\]

Figure 2: The ideal analog-to-digital converter.

- **Sampling Rate**: number of samples taken per second (Hz);
  - \(f_s = 48\) kHz (professional studio)
  - \(f_s = 44.1\) kHz (CD)
  - \(f_s = 32\) kHz (broadcasting)

- **Sampling Period**: time interval (in seconds) between samples:
  \[T_s = \frac{1}{f_s}\] seconds.
Sampled Sinusoids

- Sampling corresponds to transforming the continuous time variable $t$ into a set of discrete times that are integer $n$ multiples of the sampling period $T_s$:

$$t \rightarrow nT_s,$$

- Integer $n$ corresponds to the index in the sequence.

- Continuous sinusoid:

$$x(t) = A \sin(\omega t + \phi).$$

- Discrete sinusoid

$$x(n) = A \sin(\omega nT_s + \phi),$$

a sequence of numbers that may be indexed by $n$. 
Question I

• If the following sinusoid was sampled at $f_s = 16$,
  - what is the duration shown?
  - what is the sinusoid’s frequency?

• What would the duration be if $f_s = 32$?
Answer 1

- If $f_s = 16$ then the **sampling period** (time between samples) is

$$T_s = \frac{1}{16} \text{ s}.$$ 

- Since the signal is 24 samples long, the **duration** is

$$24 \times T_s = \frac{24}{16} = \frac{3}{2} = 1.5 \text{ s}.$$ 

- Since 16 samples correspond to 1 second AND there are 2 cycles in 1 second, the **frequency** is 2 Hz.

- Alternatively, 1 period has duration (period)

$$T = 8 \times T_s = \frac{8}{16} = \frac{1}{2}$$

and frequency of $1/T = 2 \text{ Hz}$. 

![Sample Index](Image)
• If the sampling rate is $f_s = 32$, then the duration is

$$24 \times T_s = \frac{24}{32} = \frac{3}{4} = 0.75 \text{ s},$$

and the sinusoid’s frequency is 4 Hz.
Sampling and Reconstruction

- Once $x(t)$ is sampled to produce $x(n)$, *time scale information is lost.*
- $x(n)$ may represent a number of possible waveforms.

- Reconstructing at half the sampling rate ($F_s/2$) will double the time between samples ($2/F_s$), making the sinusoid twice as long and halving the frequency.
Importance of Knowing Sampling Rate

- If the signal is digitized and reconstructed using the same sampling rate, the frequency and duration will be preserved.

- If reconstruction is done using a different sampling rate, it will change the
  - time interval between samples (signal duration),
  - time to complete one cycle (signal frequency).
Questions II

• A 220 Hz sinusoid is sampled at 44100. It is played on an audio system having a sampling rate of 22050. At what frequency will it sound?

• What is the sampling rate and frequency of the following sinusoid?
If a 220 Hz sinusoid is originally sampled at 44100 and played using a sampling rate of 22050 (half the original rate) then the period between the samples will be twice as long,

\[ T_s = \frac{1}{22050} = \frac{2}{44100}. \]

This has the effect of doubling the period of oscillation resulting in a sounding frequency that is halved (or an octave lower).

The sinusoid has 8 samples in 1 second, and thus \( f_s = 8 \). The period has 16 samples and is thus \( 16 \times 1/8 = 2 \) seconds long. The frequency is thus .5 Hz.
Implications of Sampling

- Is a sampled sequence only an approximation of the original?
- Is it possible to perfectly reconstruct a sampled signal?
- Will anything less than an infinite sampling rate introduce error?
- How frequently must we sample in order to “faithfully” reproduce an analog waveform?
Nyquist Sampling Theorem

- The **Nyquist Sampling Theorem** states that:
  
  A bandlimited continuous-time signal can be sampled and perfectly reconstructed from its samples if the waveform is sampled over twice as fast as it’s highest frequency component.

- **Nyquist limit**: the highest frequency component that can be accurately represented:
  
  \[ f_{\text{max}} < \frac{f_s}{2}. \]

- **Nyquist frequency**: sampling rate required to accurately represent up to \( f_{\text{max}} \):
  
  \[ f_s > 2f_{\text{max}}. \]

- No information is lost if sampling above \( 2f_{\text{max}} \).
- No information is gained by sampling much faster than \( 2f_{\text{max}} \).

- Is \( f_s = 44,100 \) Hz (CD-quality) enough?
Digital Audio System

- Low-pass filter (left) prevents frequencies higher than $f_s/2$ from being seen by the ADC.

- COMPUTER processing may introduce components where $f_{\text{max}} > f_s/2$.

- Low-pass filter (right) does not prevent introduction of frequencies higher than $f_s/2$!
  
  – rather, ensures they are NOT played out the DAC.

- So what exactly happens to the frequency components that exceed the Nyquist limit ($f_s/2$)?
Undersampling

- If a signal is undersampled, it will be interpreted as the alias lying in the permitted range ($f < f_s/2$);

![Graph showing undersampling](image)

Figure 3: Undersampling a 3 Hz sinusoid causes its frequency to be interpreted as 1 Hz.
What is an Alias?

- It may be shown mathematically that all discrete sinusoids have an infinite number of aliases.

- Infinite sinusoids (having different frequencies) can be reconstructed from the sinusoid sampled at $f_s$.

- A sinusoid at frequency $f_0$

  $$x(n) = A \cos(2\pi f_0 n T_s + \phi)$$

  is indistinguishable from one at frequency $f_0 + lf_s$,

  $$y(t) = A \cos(2\pi (f_0 \pm lf_s) n T_s + \phi),$$

  where $l$ is an integer.
Aliasing / Folding over

• A signal exceeding Nyquist limit $f_s/2$ will have a negative frequency component with an alias falling within the sounding bandwidth (the shaded area).

• Any signal above the Nyquist limit will be interpreted as its alias having frequency below $f_s/2$. 
Folding Frequency (Nyquist Limit)

- Let $f_{\text{in}}$ be the input signal and $f_{\text{out}}$ be the signal at the output (after the lowpass filter).
- If $f_{\text{in}}$ is less than the Nyquist limit, 
  
  $$f_{\text{out}} = f_{\text{in}},$$

  otherwise there is a folding over $f_s/2$.

Figure 4: Folding of a sinusoid sampled at $f_s = 2000$ samples per second.

- The folding occurs because of aliases of the negative frequency components.
In the Beatles track “In My Life” (1:28) there is a Baroque-style piano solo composed and played by George Martin:

- piano solo

**Problem:** George Martin’s couldn’t play the solo fast enough.

**Solution:** Play it an octave lower and at half tempo and record at 1/2 sampling rate, then play it back at twice the rate at which it was recorded.

**Effect:** The distortion creates a sort of harpsichord sound at the regular tempo and pitch.

Listen to how the original sound regains a piano quality when played at 1/2 the sampling rate:

- piano solo at 1/2 sampling rate
Quantization

- Where sampling is the process of taking a sample at regular time intervals...

- **Quantization** is the process of assigning a value (from a finite number of possibilities) to the amplitude of the signal at a time sample.

- Computers use bits to store such data.

- The higher the number of bits used to represent a value, the greater the number of possible values and the more precise the sampled amplitude will be.

- With $n$ bits, $2^n$ possible values that can be represented.

- For CD quality audio, the number of bits is $n = 16$:
  - each sample can have $2^{16} = 65,536$ possible values;
  - the highest possible amplitude is $2^{15} = 32,768$, (since audio signals are positive and negative).

- Does quantization introduce error?
Quantization (linear)

- The signal below has 11 possible values to which the instantaneous amplitude may be quantized.
Quantization Error

• In a linear converter, when the actual sample value falls between these values (solid blue lines), it is quantized (or rounded) to the nearest value.

• This introduces a quantization error that will be uniformly distributed between 0 and 1/2 (error will never be greater than a factor of 1/2 the increment).
Bit Depth

- Computers use bits to store sample values—the number of bits used is called the **bit depth**: the greater the bit depth,
  - the greater the number of possible values,
  - the more precise the sampled amplitude will be.

- With \( n \) bits, \( 2^n \) possible values that can be represented.
  - for CD quality audio, \( n = 16 \),
  - each sample can have \( 2^{16} = 65,536 \) possible values,
  - the highest possible amplitude is \( 2^{15} = 32,768 \), (since audio signals are positive and negative).
Signal to Quantization Error (SQNR)

- When noise is a result of quantization error, we determine its audibility using the signal-to-quantization-noise-ratio (SQNR), determined by the ratio of
  - the maximum amplitude \(2^{n-1}\) to
  - the maximum quantization noise (1/2 for a linear converter).

- To determine audibility, the SQNR is provided in decibels (dB):
  
  \[
  20 \log_{10} \left( \frac{2^{n-1}}{1/2} \right) \, \text{dB} = 20 \log_{10}(2^n) \, \text{dB} = n \times 20 \log_{10}(2) \, \text{dB} \approx n \times 6 \, \text{dB} \approx 96 \, \text{dB (for 16 bits)}.
  \]

- Note: a sound with an amplitude 40dB below maximum would have a SQNR of only 56 dB.
Quantization Error for Synthesis

- If a signal at half maximum amplitude is present

\[
20 \log_{10}(2^n - 1) \approx (n - 1) \times 6 \text{ dB}
\]

\[
\approx 90 \text{ dB (for 16 bits)}
\]

- Though 16-bits is usually considered acceptable for representing audio with good SNQR, it's when we begin processing the sound that error compounds.

- For this reason, software such as Pd will actually use 32 (floating point) or 64 bits (double-precision floating point) to represent a value.