Analog Signals

- A signal, of which a sinusoid is only one example, is a set, or sequence of numbers.
- The term “analog” refers to the fact that it is “analogous” of the signal it represents.
- A "real-world" signal is captured using a microphone which has a diaphragm that is pushed back and forth according to the compression and rarefaction of the sounding pressure waveform.

![Figure 1: The electrical signal used to represent the pressure variations of a sound wave is an analog signal.]

- The microphone transforms this displacement into a time-varying voltage—an analog electrical signal.

Continuous-Time Signals

- Analog signals are *continuous* in time.
- A *continuous-time* signal is an *infinite* and *uncountable* sequence of numbers, as are the possible values each number can have.
  - that is, between a start and end time, there are infinite possible values for time \( t \) and instantaneous amplitude \( x(t) \).
- A continuous signal cannot be stored, or processed, in a computer since it would require an infinite amount of data.
- Analog signals must therefore be *discretized*, or *digitized*, to produce a a finite set of numbers, for computer use.

Discrete-Time, Digital Signals

- When analog signals are brought into a computer, they must be made *discrete*.
- In a *discrete-time signal*, the number of elements in the set, as well as the possible values of each element, is finite, countable, and can be represented with computer bits, and stored on a digital storage medium.

- **Sampling**: the process of taking individual values of a continuous-time signal (at regular time intervals).
Analog to Digital Conversion

- The process by which an analog signal is digitized is called analog-to-digital or “a-to-d” conversion
- analog-to-digital converter (ADC): the device that digitizes (discretizes) an analog signal.
- The ADC must accomplish two (2) tasks:
  1. sampling:
     - take values at regular time intervals;
  2. quantization:
     - assign a number to the value (using limited computer bits).

Sampling

- Sampling: process of taking sample values of the continuous waveform at regularly spaced time intervals.

\[ x[n] = x(nT_s) \]

\[ T_s = 1/f_s \]

Figure 2: The ideal analog-to-digital converter.

- Sampling Rate: number of samples taken per second (Hz):
  - \( f_s = 48 \text{ kHz} \) (professional studio)
  - \( f_s = 44.1 \text{ kHz} \) (CD)
  - \( f_s = 32 \text{ kHz} \) (broadcasting)

- Sampling Period: time interval (in seconds) between samples:
  \[ T_s = 1/f_s \text{ seconds} \]

Sampled Sinusoids

- Sampling corresponds to transforming the continuous time variable \( t \) into a set of discrete times that are integer \( n \) multiples of the sampling period \( T_s \):
  \[ t \rightarrow nT_s, \]
- Integer \( n \) corresponds to the index in the sequence.
- Continuous sinusoid:
  \[ x(t) = A \sin(\omega t + \phi). \]
- Discrete sinusoid
  \[ x(n) = A \sin(\omega nT_s + \phi), \]
  a sequence of numbers that may be indexed by \( n \).

Question I

- If the following sinusoid was sampled at \( f_s = 16 \),
  - what is the duration shown?
  - what is the sinusoid’s frequency?

- What would the duration be if \( f_s = 32 \)?
Answer 1

- If $f_s = 16$ then the **sampling period** (time between samples) is
  $$T_s = \frac{1}{16} \text{s}.$$  

- Since the signal is 24 samples long, the **duration** is
  $$24 \times T_s = \frac{24}{16} = \frac{3}{2} = 1.5 \text{s}.$$  

- Since 16 samples correspond to 1 seconds AND there are 2 cycles in 1 second, the **frequency** is 2 Hz.

- Alternatively, 1 period has duration (period)
  $$T = 8 \times T_s = \frac{8}{16} = \frac{1}{2}$$  
and frequency of $1/T = 2 \text{ Hz}.$

- If the sampling rate is $f_s = 32$, then the duration is
  $$24 \times T_s = \frac{24}{32} = \frac{3}{4} = 0.75 \text{ s},$$  
and the sinusoid’s frequency is 4 Hz.

Sampling and Reconstruction

- Once $x(t)$ is sampled to produce $x(n)$, **time scale information is lost**.

- $x(n)$ may represent a number of possible waveforms.

- **Reconstructing at half the sampling rate** ($f_s/2$) will double the time between samples ($2/f_s$), making the sinusoid **twice as long** and **halving the frequency**.

Importance of Knowing Sampling Rate

- If the signal is digitized and reconstructed **using the same sampling rate**, the frequency and duration will be preserved.

- If reconstruction is done **using a different sampling rate**, it will change the
  - time interval between samples (signal **duration**),
  - time to complete one cycle (signal **frequency**).
Questions II

• A 220 Hz sinusoid is sampled at 44100. It is played on an audio system having a sampling rate of 22050. At what frequency will it sound?
• What is the sampling rate and frequency of the following sinusoid?

![Graph of a sampled sinusoid]

Answers II

• If a 220 Hz sinusoid is originally sampled at 44100 and played using a sampling rate of 22050 (half the original rate) then the period between the samples will be twice as long,

\[ T_s = \frac{1}{22050} = \frac{2}{44100} \]

This has the effect of doubling the period of oscillation resulting in a sounding frequency that is halved (or an octave lower).

![Graph showing sampled sinusoid]

• The sinusoid has 8 samples in 1 second, and thus \( f_s = 8 \). The period has 16 samples and is thus \( 16 \times 1/8 = 2 \) seconds long. The frequency is thus .5 Hz.

Implications of Sampling

• Is a sampled sequence only an approximation of the original?
• Is it possible to perfectly reconstruct a sampled signal?
• Will anything less than an infinite sampling rate introduce error?
• How frequently must we sample in order to “faithfully” reproduce an analog waveform?

Nyquist Sampling Theorem

• The Nyquist Sampling Theorem states that:

  A bandlimited continuous-time signal can be sampled and perfectly reconstructed from its samples if the waveform is sampled over twice as fast as it’s highest frequency component.

  • Nyquist limit: the highest frequency component that can be accurately represented:
  
  \[ f_{\text{max}} < \frac{f_s}{2} \]

  • Nyquist frequency: sampling rate required to accurately represent up to \( f_{\text{max}} \):
  
  \[ f_s > 2f_{\text{max}} \]

  • No information is lost if sampling above \( 2f_{\text{max}} \).
  • No information is gained by sampling much faster than \( 2f_{\text{max}} \).
  • Is \( f_s = 44,100 \) Hz (CD-quality) enough?
Digital Audio System

• Low-pass filter (left) prevents frequencies higher than \( f_s/2 \) from being seen by the ADC.
• COMPUTER processing may introduce components where \( f_{\text{max}} > f_s/2 \).
• Low-pass filter (right) does not prevent introduction of frequencies higher than \( f_s/2 \)!
  — rather, ensures they are NOT played out the DAC.
• So what exactly happens to the frequency components that exceed the Nyquist limit \( (f_s/2) \)?

Undersampling

• If a signal is undersampled, it will be interpreted as the alias lying in the permitted range \( (f < f_s/2) \);

What is an Alias?

• It may be shown mathematically that all discrete sinusoids have an infinite number of aliases.

Aliasing / Folding over

• A signal exceeding Nyquist limit \( f_s/2 \) will have a negative frequency component with an alias falling within the sounding bandwidth (the shaded area).
• Any signal above the Nyquist limit will be interpreted as its alias having frequency below \( f_s/2 \).
Folding Frequency (Nyquist Limit)

- Let \( f_{in} \) be the input signal and \( f_{out} \) be the signal at the output (after the lowpass filter).
- If \( f_{in} \) is less than the Nyquist limit,
  \[ f_{out} = f_{in} \]
  otherwise there is a folding over \( f_s/2 \).

Figure 4: Folding of a sinusoid sampled at \( f_s = 2000 \) samples per second.

- The folding occurs because of aliases of the negative frequency components.

Quantization

- Where sampling is the process of taking a sample at regular time intervals...
- Quantization is the process of assigning a value (from a finite number of possibilities) to the amplitude of the signal at a time sample.
- Computers use bits to store such data.
- The higher the number of bits used to represent a value, the greater the number of possible values and the more precise the sampled amplitude will be.
- With \( n \) bits, \( 2^n \) possible values that can be represented.
- For CD quality audio, the number of bits is \( n = 16 \):
  - each sample can have \( 2^{16} = 65,536 \) possible values;
  - the highest possible amplitude is \( 2^{15} = 32,768 \), (since audio signals are positive and negative).
- Does quantization introduce error?

Sampling in Practice II

- In the Beatles track “In My Life” (1:28) there is a Baroque-style piano solo composed and played by George Martin:
  - piano solo
- **Problem:** George Martin’s couldn’t play the solo fast enough.
- **Solution:** Play it an octave lower and at half tempo and record at 1/2 sampling rate, then play it back at twice the rate at which it was recorded.
- **Effect:** The distortion creates a sort of harpsichord sound at the regular tempo and pitch.
  Listen to how the original sound regains a piano quality when played at 1/2 the sampling rate:
  - piano solo at 1/2 sampling rate

Quantization (linear)

- The signal below has 11 possible values to which the instantaneous amplitude may be quantized.
Quantization Error

- In a linear converter, when the actual sample value falls between these values (solid blue lines), it is quantized (or rounded) to the nearest value.
- This introduces a quantization error that will be uniformly distributed between 0 and 1/2 (error will never be greater than a factor of 1/2 the increment).

Bit Depth

- Computers use bits to store sample values—the number of bits used is called the bit depth: the greater the bit depth,
  - the greater the number of possible values,
  - the more precise the sampled amplitude will be.
- With $n$ bits, $2^n$ possible values that can be represented.
  - for CD quality audio, $n = 16$,
  - each sample can have $2^{16} = 65,536$ possible values,
  - the highest possible amplitude is $2^{15} = 32,768$, (since audio signals are positive and negative).

Signal to Quantization Error (SQNR)

- When noise is a result of quantization error, we determine its audibility using the signal-to-quantization-noise-ratio (SQNR), determined by the ratio of
  - the maximum amplitude ($2^{n-1}$) to
  - the maximum quantization noise (1/2 for a linear converter).
- To determine audibility, the SQNR is provided in decibels (dB):
  $$20 \log_{10} \left( \frac{2^{n-1}}{1/2} \right) \text{ dB} = 20 \log_{10} (2^n) \text{ dB} = n \times 20 \log_{10}(2) \text{ dB} \approx n \times 6 \text{ dB} \approx 96 \text{ dB (for 16 bits)}.$$
- Note: a sound with an amplitude 40dB below maximum would have a SQNR of only 56 dB.

Quantization Error for Synthesis

- If a signal at half maximum amplitude is present
  $$20 \log_{10}(2^{n-1}) \approx (n - 1) \times 6 \text{ dB} \approx 90 \text{ dB (for 16 bits)}$$
- Though 16-bits is usually considered acceptable for representing audio with good SNQR, its when we begin processing the sound that error compounds.
- For this reason, software such as Pd will actually use 32 (floating point) or 64 bits (double-precision floating point) to represent a value.