Music 206: Digital Waveguides

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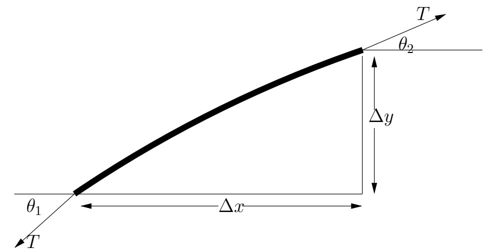
Motion for a Wave

- The 1-dimensional digital waveguide model is a discrete-time solution to the 1-dimensional wave equation.
- To derive the equation for the *transverse displacement wave* on a string, consider a small section of a string with mass
$$m = \mu \Delta x,$$
where μ is the mass per unit length.



- The net *vertical force* on this section is the difference between the y components of the tension T[1]:
$$F_n = T \sin\theta_2 - T \sin\theta_1.$$

[1] Tension is the magnitude of the force due to stretching the string

Newton's Law

- The slope of the string at each end of the section is given by
$$\mathsf{m}_1 = \frac{\partial y_1}{\partial x} = \tan\theta_1$$
$$\mathsf{m}_2 = \frac{\partial y_2}{\partial x} = \tan\theta_2$$
- If the displacement of the string from equilibrium is small, then angles θ_1 and θ_2 are small, and
$$\sin\theta_1 \approx \tan\theta_1 \quad \text{and} \quad \sin\theta_2 \approx \tan\theta_2$$
- The net vertical force can therefore be written as
$$F_n = T \sin\theta_2 - T \sin\theta_1 = T\left(\mathsf{m}_2 - \mathsf{m}_1\right) = T \Delta \mathsf{m}.$$
- By Newton's second law, $F = ma$,
$$T \Delta \mathsf{m} = (\mu \Delta x) \left(\frac{\partial^2 y}{\partial t^2}\right)$$
$$T \frac{\Delta \mathsf{m}}{\Delta x} = \frac{\mu \partial^2 y}{\partial t^2},$$
where acceleration is given by $a = \partial^2 y / \partial t^2$.

The 1-D wave equation

- Taking the limit as $\Delta x \longrightarrow 0$
$$\lim_{\Delta x \to 0} \frac{\Delta \mathsf{m}}{\Delta x} = \frac{\partial \mathsf{m}}{\partial x} = \frac{\partial^2 y}{\partial x^2},$$
we obtain the one-dimensional wave equation given by
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$
where $c = \sqrt{T/\mu}$ is the speed of wave propagation.

Travelling Wave Solution

- The general class of solutions to the lossless, one-dimensional, second-order wave equation can be expressed as

$$y(t,x) = y_r\left(t - \frac{x}{c}\right) + y_l\left(t + \frac{x}{c}\right).$$

where

$$\begin{aligned} y_r(t - x/c) &\triangleq \text{right-going traveling waves} \\ y_l(t + x/c) &\triangleq \text{left-going traveling waves} \end{aligned}$$

and where y_r and y_l are assumed twice-differentiable.

- This **traveling-wave solution** of the wave equation was first published by d'Alembert in 1747.
- Notice the traveling-wave solution of the 1-D wave equation has replaced a function of two variables $y(t,x)$, by two functions of a single variable in time units[2], greatly reducing computational complexity.

[2] If x is in meters and sound velocity c is in meters per second, x/c is in seconds and the spatial variable x cancels out.

Sampled Travelling Waves

- Sampling is carried out by the change of variables

$$\begin{aligned} x \rightarrow x_m &= mX \\ t \rightarrow t_n &= nT, \end{aligned}$$

where T is the temporal sampling interval, and $X \triangleq cT$, is the spatial sampling interval

- Substituting into the traveling-wave solution yields

$$\begin{aligned} y(t_n, x_m) &= y_r(t_n - x_m/c) + y_l(t_n + x_m/c) \\ &= y_r(nT - mX/c) + y_l(nT + mX/c) \\ &= y_r\left[(n-m)T\right] + y_l\left[(n+m)T\right] \end{aligned}$$

- Since T multiplies all arguments it is typically suppressed.
- This notation is also used to model pressure in acoustic tubes.

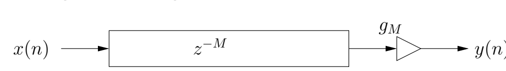
Traveling Waves

- A traveling wave is any kind of wave that propagates in a single direction with negligible change in shape.
- A delayline (pure delay) can model wave propagation with a fixed waveshape in 1-D.
- **Transverse and longitudinal** waves[3] in a vibrating string are nearly perfect traveling waves.
- **Plane waves** are a class of traveling wave that dominate in cylindrical bores (bore of clarinet, cylindrical tube segments in trumpet).
- **Spherical waves** take the place of plane waves in conical tubes. Because they travel like plane waves, they are still modeled with a delay line.

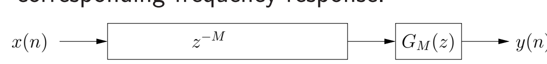
[3] Recall, transverse and longitudinal waves are waves in which the partical displacement is perpendicular and parallel, respectively, to the direction of the traveling wave.

Damped Travelling Waves

- When travelling waves are damped, their losses are **distributed** along the length of the system.
- Rather than applying losses at each time-step, they may be accumulated, or **lumped**, at discrete points along the delay line.
- If the losses are the same for each frequency, they may be simulated using a simple scaling of the delay line input or output.



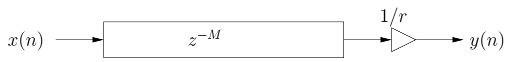
- If losses are frequency dependent they are implemented using a digital filter $G(z)$ with the corresponding frequency response.



- The input-output simulation is exact, while the signal samples inside the delay line have a slight gain error.
- If the internal signals are needed later, they can be tapped out using correcting gains relative to the tapping location.

Losses due to spherical spreading

- In a spherical pressure wave of radius r, the energy of the wavefront is spread out over surface area $4\pi r^2$.
- The intensity of an expending sphererical wave decreases as $1/r^2$.
- This **spherical spreading loss**, exemplifies an inverse square law.
- Sound pressure amplitude of a traveling wave is proportional to the square-root of its energy/intensity, leading to an amplitude proportional to $1/r$ for spherical traveling waves.



- Though delay lines may be used for spherical waves of radius r, there is a gain of $1/r$ applied to the output/input.
- Waves propagating a distance r_0 from the cone apex to a disance r_1 from the cone apex, will experience a pressure scaling of r_0/r_1.

Converting Propagation Distance to Delay Length

- We may regard the delay-line memory itself as the fixed "air" that propagates sound samples at a fixed speed c.
- The number of delay samples is the propagation distance divided by the distance sound propagates in one sample.
- If the listening point is d meters away from the source, then the delay line length M needs to be

$$M = \frac{d}{X} = \frac{d}{cT} \text{ samples.}$$

Reflection of Spherical or Plane Waves

- When a wave reaches a wall or other obstacle, it is either *reflected* or *scattered*:
  - *reflection* occurs when the surface is flat for plane waves, or curved with the appropriate radius (for spherical waves);
  - *scattering* occurs when the surface has variations on the scale of the spatial wavelength.

Absorption

- In air, there is always significant additional (frequency-dependent) loss caused by air absorption.
- Wave propagation in vibrating strings undergoes an analogous absorption loss, as does the propagation of nearly every other kind of wave in the physical world.

Reflection at a Fixed End of String

- Consider the displacement of a string at the boundary $x = 0$:

$$\begin{aligned} y &= y_r(t - 0/c) + y_l(t + 0/c) \\ &= y_r(t) + y_l(t). \end{aligned}$$

- If **fixed**, it's displacement y is zero:

$$\begin{aligned} y_r(t) + y_l(t) &= 0 \\ y_r(t) &= -y_l(t). \end{aligned}$$

the reflected wave is equal but opposite to the right traveling wave.

- See animation: Reflection from a fixed boundary

Reflection at a Free End of the String

- At $x = 0$, if a string is free there is **no transverse force**.
- Net transverse force is proportional to the slope $\partial y/\partial x$:

$$\begin{aligned}\frac{\partial y}{\partial x} &= \frac{\partial}{\partial x} y_r(t - x/c) + \frac{\partial}{\partial x} y_l(t + x/c) \\ &= \frac{1}{c}\left[y_l(t + 0/c) - y_r(t - 0/c)\right] \\ &= 0 \\ y_r(t) &= y_l(t).\end{aligned}$$

> The reflected wave is equal to the incident wave with no change of sign.

- See animation: Reflection from a free boundary

Standing Waves

- Reflection causes destructive and constructive interference, leading to standing waves.
- **Standing Waves**:
  - created by the sum or right and left traveling waves:
    standing wave animation
  - it is a pattern of alternating nodes and antinodes
    nodes and antinodes animation
  - the fundamental mode of oscillation, is determined by the *shortest* node-antinode pattern.
- Standing waves created from a fixed boundary:
  - animation of standing waves, fixed
- Standing waves created from a free boundary:
  - animation standing waves, free

Guitar String

- The guitar string is fixed at both ends—therefore it has a node at both ends.

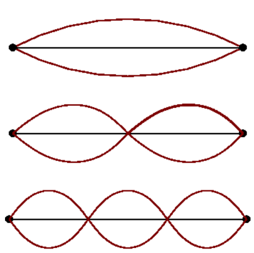


Figure 1: Standing waves on a guitar string.

- For string length L and each harmonic number n,
  - the wavelength is $\lambda_n = \frac{2}{n}L$.
  - the fundamental frequency is $f_n = \frac{c}{\lambda_n} = n\frac{c}{2L} = nf_1$.

Wave Impedance

- The **wave impedance**
  - represents the medium's **resitance** to distortion in the presence of an external force;
  - a characteristic of the medium in which a wave propagates and given by:

$$Z = \frac{F}{v} = \frac{p}{U},$$

$$\begin{aligned} v &\triangleq \text{particle velocity;} \\ p = F/A &\triangleq \text{acoustic pressure;} \\ U = v/A &\triangleq \text{volume velocity (airflow);} \\ A &\triangleq \text{medium cross-sectional area.}\end{aligned}$$

- **For solids**,

$$Z_0 = \rho c,$$

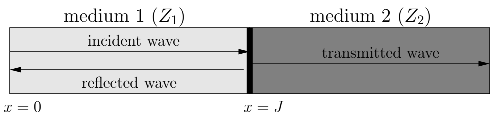
where ρ is the density of the material.

- **For gases** in cylindrical tubes with a cross sectional area of A,

$$Z_0 = \frac{\rho c}{A}.$$

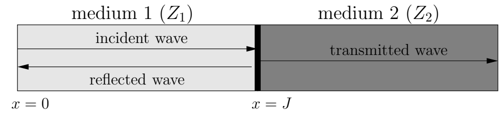
Reflection due to Change of Impedance

- It is possible to have a boundary which is neither completely fixed nor completely free.
- When a wave traveling in a medium with impedance Z_1 confronts a new medium with impedance Z_2,
  - incident wave will be partially reflected;
  - amplitude and polarity of reflected wave dependent on the impedance of the two mediums.



- What is not reflected at the boundary is transmitted through to the new medium.

Boundary Conditions



- The pressure and velocity at the junction must be equal (continuity—they are shared by both mediums).
- **The pressure** at the junction is given by

$$p(J) = p_i(J) + p_r(J),$$

$$p_i(J) \triangleq \text{the indicident wave at the junction}$$
$$p_r(J) \triangleq \text{the reflected wave at the junction.}$$

- The incident and reflected pressure waves correspond to incident and reflected airflow U by

$$p_i(J) = Z_1 U_i(J)$$

and

$$p_r(J) = -Z_1 U_r(J),$$

  - the negative sign accounts for the fact that airflow is a directional quantity and moves in the direction in which it generates pressure.

Volume Velocity at Junction

- **Volume velocity** at the junction is the sum of the incident and reflected velocity waves:

$$U(J) = U_i(J) + U_r(J).$$

- Incorporating the result for the incident pressure wave:

$$p_i(J) = Z_1 U_i(J),$$

and reflected pressure wave:

$$p_r(J) = -Z_1 U_r(J),$$

volume velocity at the junction is

$$U(J) = \frac{1}{Z_1}(p_i(J) - p_r(J)).$$

Calculating reflection coefficient (pressure)

- The new medium at the junction has impedance

$$\begin{aligned} Z_2 &= \frac{p(J)}{U(J)} \\ &= \frac{p_i(J) + p_r(J)}{(p_i(J) - p_r(J))/Z_1} \\ &= Z_1 \left(\frac{p_i(J) + p_r(J)}{p_i(J) - p_r(J)} \right). \end{aligned}$$

- This result can be used to calculate the reflection coefficient p_r/p_i:

$$\begin{aligned} Z_1 \frac{p_i + p_r}{p_i - p_r} &= Z_2 \\ Z_1(p_i + p_r) &= Z_2(p_i - p_r) \\ p_r(Z_2 + Z_1) &= p_i(Z_2 - Z_1) \\ \frac{p_r}{p_i} &= \frac{Z_2 - Z_1}{Z_2 + Z_1}. \end{aligned}$$

Calculating reflection coefficient (velocity)

- Likewise, the new impedance Z_2 may be given in terms of velocity components

$$Z_2 = \frac{p(J)}{U(J)} = Z_1 \frac{U_i(J) - U_r(J)}{U_i(J) + U_r(J)}$$

- The reflection coefficient is calculated as:

$$\begin{aligned} Z_1 \frac{U_i - U_r}{U_i + U_r} &= Z_2 \\ Z_1(U_i - U_r) &= Z_2(U_i + U_r) \\ U_i(Z_1 - Z_2) &= U_r(Z_1 + Z_2) \\ \frac{U_r}{U_i} &= \frac{Z_1 - Z_2}{Z_1 + Z_2}. \end{aligned}$$

Applying to Reflections on String

- Consider a string **free at one end** ($Z_2 = 0$).
  - Using displacement velocity as the wave variable:

$$\frac{v_r}{v_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{Z_1}{Z_1} = 1,$$

reflected velocity wave = incident velocity wave.

  - The total velocity at the junction (boundary) is

$$v = v_i + v_r = v_i + v_i = 2v_i.$$

- If the string is **fixed at one end**, new impedance approaches infinity ($Z_2 \gg Z_1$):

$$\frac{v_r}{v_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \approx -\frac{Z_2}{Z_2} = -1,$$

where Z_1 is considered to be negligible.

- The velocity at the boundary is therefore

$$v = v_i + v_r = v_i - v_i = 0.$$

Applying to Reflection in a Tube

- Recall that impedance for air in a confined space is inversely proportional to it's area (the smaller the area, the larger the impedance).
- For a tube **open at one end**
  - assume the wave impedance of open air is negligible to that of the tube cylindrical section:
  - the reflection coefficient for pressure at an open end is

$$\frac{p_r}{p_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = -\frac{Z_1}{Z_1} = -1.$$

- For a tube **closed at one end**
  - assume an infinitely small radius and a corresponding wave impedance Z_2 that is very large relative to Z_1:

$$\frac{p_r}{p_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{Z_2}{Z_2} = 1.$$

Wave Variables

- Displacement, velocity and acceleration waves all reflect with the same polarity:

$$r_{x,v,a} = \frac{Z_1 - Z_2}{Z_1 + Z_2}.$$

- Force and pressure waves however, reflect with an opposite polarity:

$$r_{f,p} = \frac{Z_2 - Z_1}{Z_2 + Z_1}.$$

The Plucked String

- When a string is plucked, the finger chooses a point on the string and then displaces it a certain distance.
- Plucking a string therefore, introduces an initial energy displacement (potential energy).
- The shape of the string before its release defines which harmonics will be present in the resulting motion.
- A string plucked at $1/\text{n}^{th}$ the distance from one end will not have energy at multiples of the n^{th} harmonic.
- The strength of excitation of the m^{th} vibrational mode is inversely proportional to the square of the mode number.
- A simple example is demonstrated by plucking the string exactly midway between its endpoints. In so doing, we have created a sort of triangle wave, with its corresponding harmonics.

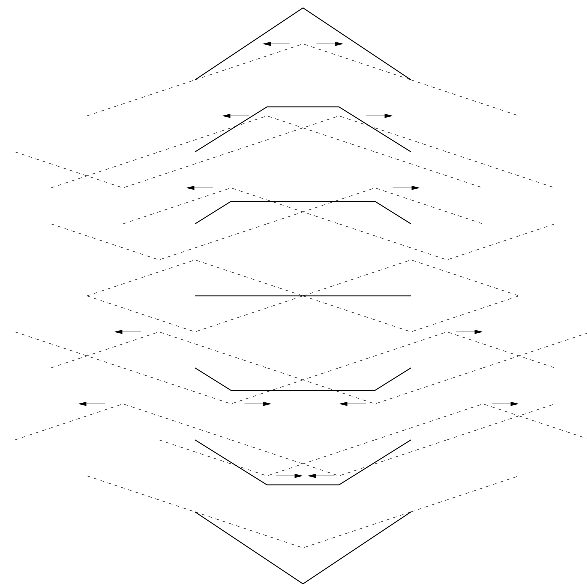


Figure 2: The motion of a string plucked one-half of the distance from one end.

String Spectrum

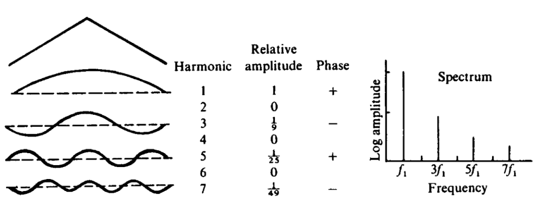


Figure 3: Spectrum of a string plucked one-half of the distance from one end.

Pluck position

- Take for example the motion of a string plucked one-fifth of the distance from the end.

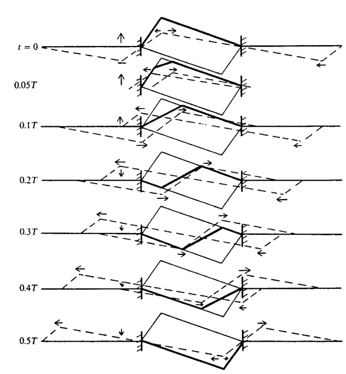


Figure 4: The motion of a string plucked one-fifth of the distance from one end.

- The motion can be thought of as two pulses moving in opposite directions (see the dashed line).
- The resulting motion consists of two bends, one moving clockwise and the other counterclockwise around a parallelogram.

Resulting Plucked String Spectrum

- The resulting spectrum of the string plucked one-fifth of the distance from one end is given below.

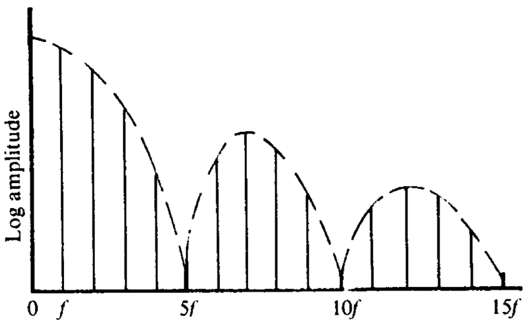
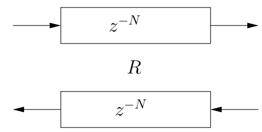


Figure 5: Spectrum of a string plucked one-fifth of the distance from one end.

Digital Waveguides

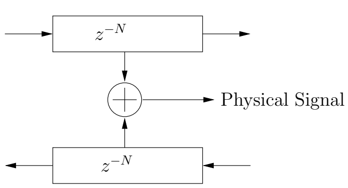
- A digital waveguide is a sampled traveling-wave simulation for waves in ideal strings or acoustic tubes.
- A (lossless) digital waveguide is defined as a bidirectional delay line at some wave impedance R.



- As before, each delay line contains a sampled acoustic traveling wave. However, since we now have a bidirectional delay line, we have two traveling waves, one to the "left" and one to the "right".
- While a single delay line can model an acoustic plane wave, a digital waveguide can model any one-dimensional linear acoustic system such as a violin string or a clarinet bore.
- In real acoustic strings and bores, the 1D waveguides exhibit some loss and dispersion so some filtering will be needed in the waveguide to obtain an accurate physical model of such systems.

Physical Outputs

- Physical variables (force, pressure, velocity, ...) are obtained by summing traveling-wave components.



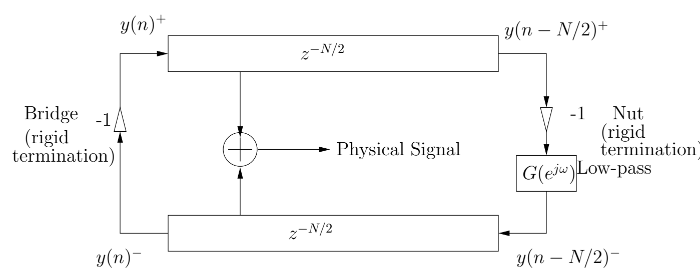
- To determine the value at any physical point, extract a physical signal from a digital waveguide using delay-line taps.
- The physical wave vibration is obtained by summing the left- and right-going traveling waves.
- The two traveling waves in a digital waveguide are now components of a more general acoustic vibration.
- A traveling wave by itself in one of the delay lines is no longer regarded as "physical" unless the signal in the opposite-going delay line is zero.

Plucked String Model

In this string simulator, there is a loop of delay containing

$$N = 2L/X = f_s/f_1 \quad \text{samples}$$

where f_1 is the desired pitch of the string and L is the physical length of the string.



- What is the impulse response of this structure?
- Define an input $x(n)$ and an output $y(n)$.