Music 206: Delay and Digital Filters I

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Digital Filters

- Any medium through which a signal passes may be regarded as a filter.
- Typically however, a filter is viewed as something which modifies the signal in some way. Examples include:
  - audio speakers / headphones
  - rooms / acoustic spaces
  - musical instruments
- A *digital* filter is a formula for going from one digital signal to another.

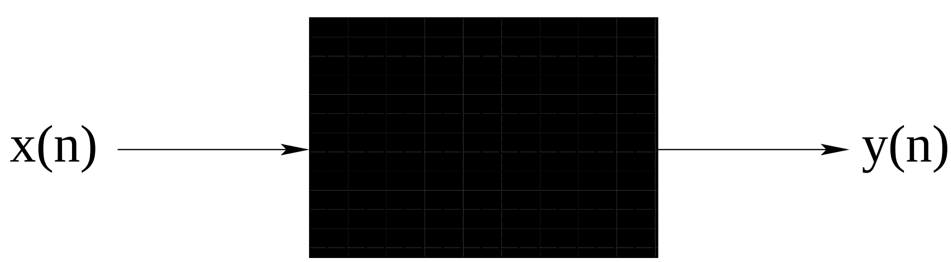


Figure 1: A black box filter.

Inside the Black Box—Pure Delay

- Time-domain implementations of digital filters involve signal *delay*, that is, *delayed* versions of input and/or output signals.
- What does it mean to delay an audio signal?
  - move it later (or earlier) in time
  - change the initial phase of signal, i.e., the value at time=0.

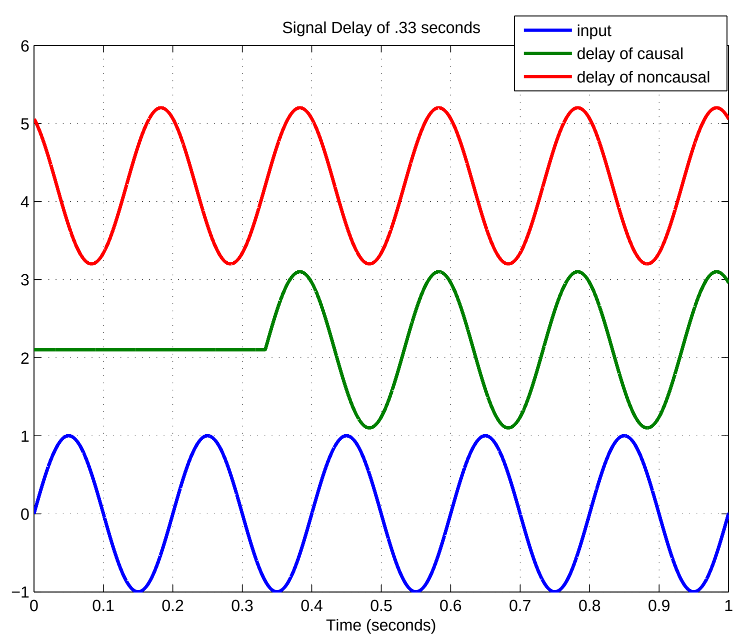
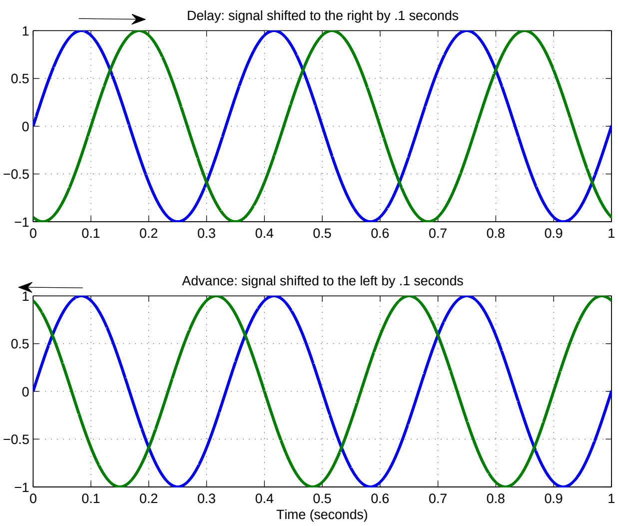


Figure 2: Timeshifting a signal will change the phase of the signal.

Time shifting a signal

- Whenever a signal can be expressed in the form
$$y(n) = x(n - M),$$
$y(n)$ is a *delayed* (time-shifted) version of $x(n)$.



- **If M is a positive number**,
$$y(n) = x(n - M),$$
the shift on the time axis is to the *right*.
- **If M is a negative number**,
$$y(n) = x(n + M),$$
the shift on the time axis is to the *left* (the signal has been advanced in time).

Time shift and addition

- Other than hearing a possible silence (for causal signals) before the signal onset, there is no audible effect of a *pure* delay.
- What happens, however, when a signal $x(n)$ is added to a delayed version of itself $x(n-M)$?
- We have created a *digital filter*, with a formula for going from input $x(n)$ to output $y(n)$ given by the difference equation

$$y(n) = x(n) + x(n-M).$$

A Running Averager

- In a simple case where $M = 1$,

$$y(n) = x(n) + x(n-1),$$

we are taking a *running average* of the input signal $x(n)$.

This filter takes the average of two adjacent samples (with a gain of 2).

Intuitive Analyis at Low Frequencies

- The running average of a signal with little or no variation from sample to sample will be very close to the input signal.
- At DC

$$x_1(n) = [A, A, A, ...].$$

- The output of the filter is

$$\begin{aligned} y(n) &= x_1(n) + x_1(n-1) \\ &= \quad [A, A, A, ...] \\ &\quad + [0, A, A, A, ...] \\ &= \quad [A, 2A, 2A, 2A, ...] \end{aligned}$$

The output is effectively the same as the input, but with a gain of 2.

Intuitive Analyis at High Frequencies

- The running average of an input signal with significant variation from sample to sample will be very different from its input.
- At $f_s/2$ (Nyquist limit)

$$x_2(n) = [A, -A, A, ...].$$

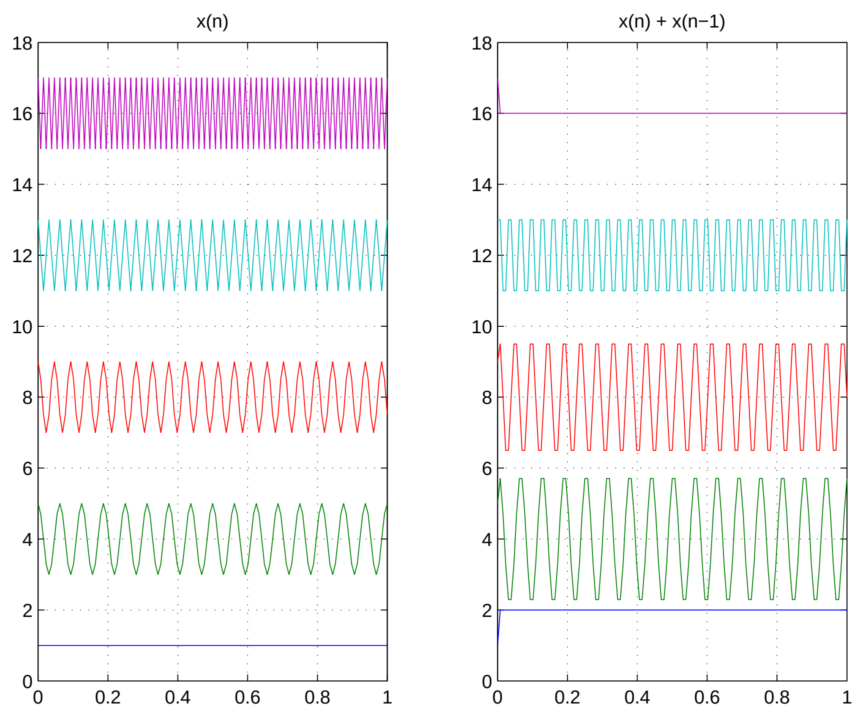
- The output of the filter is

$$\begin{aligned} y(n) &= x_2(n) + x_2(n-1) \\ &= \quad [A, -A, A, ...] \\ &\quad + [0, A, -A, A, ...] \\ &= \quad [A, 0, 0, 0, ...] \end{aligned}$$

The output is different from the input—complete attenuation.

What about all the frequencies in between?

- We may find the frequency response of the filter by checking the behaviour of the filter at every possible frequency between 0 and $f_s/2$ Hz (sinewave analysis).



- This filter boosts low frequencies while attenuating higher frequencies—it is a *lowpass* filter.

Impulse Response

- Alternatively, we can use an input signal that contains all frequency components, and then we only have to do the "checking" operation once.
- An input signal with the broadest possible spectrum would be an *impulse*.
- The response of a filter to an impulse is called an *impulse response*.

> Any filter in a large class known as linear, time-invariant (LTI), is completely characterized by its impulse response.

- What is the impulse response of this filter?

Frequency Response

- The spectrum of the impulse response gives us the *frequency response* from which we may see how the filter modifies the *amplitude* and *phase* of a signal's sinusoidal components.

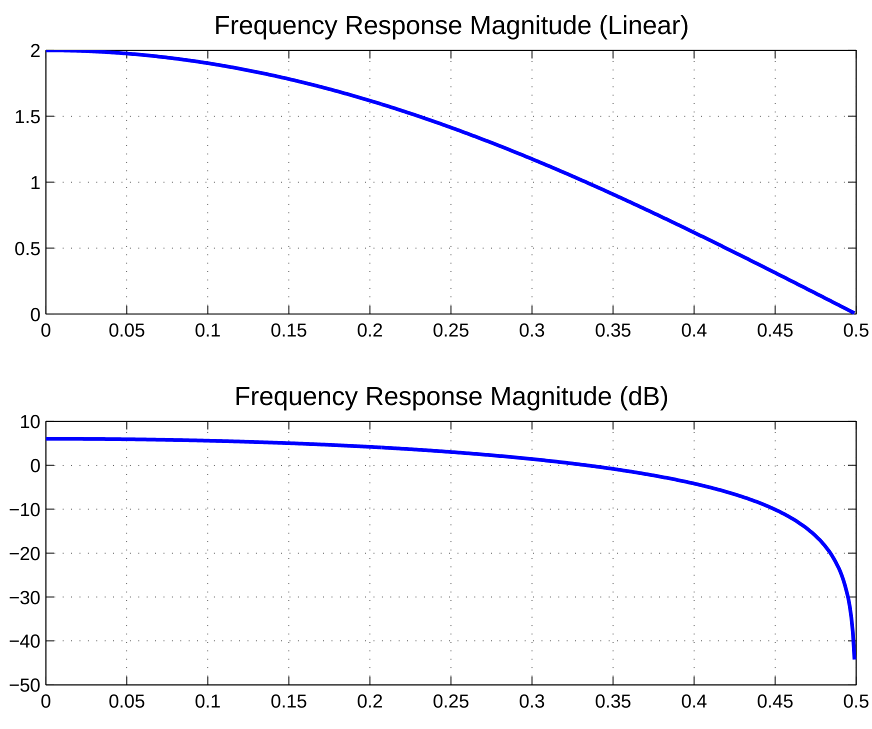


Figure 3: Magnitude of the Frequency Response shows a low-pass characteristic.

Response at the Cutoff Frequency

- Look a little closer at the filter's response to $f_s/4$.

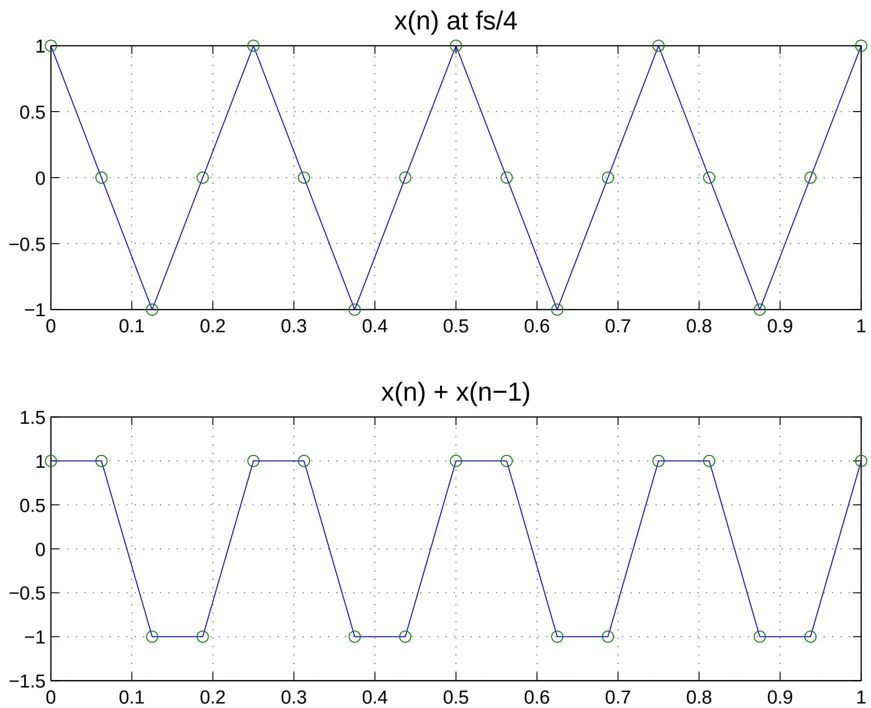


Figure 4: Filter behaviour at $f_s/4$.

Interpreting the Phase

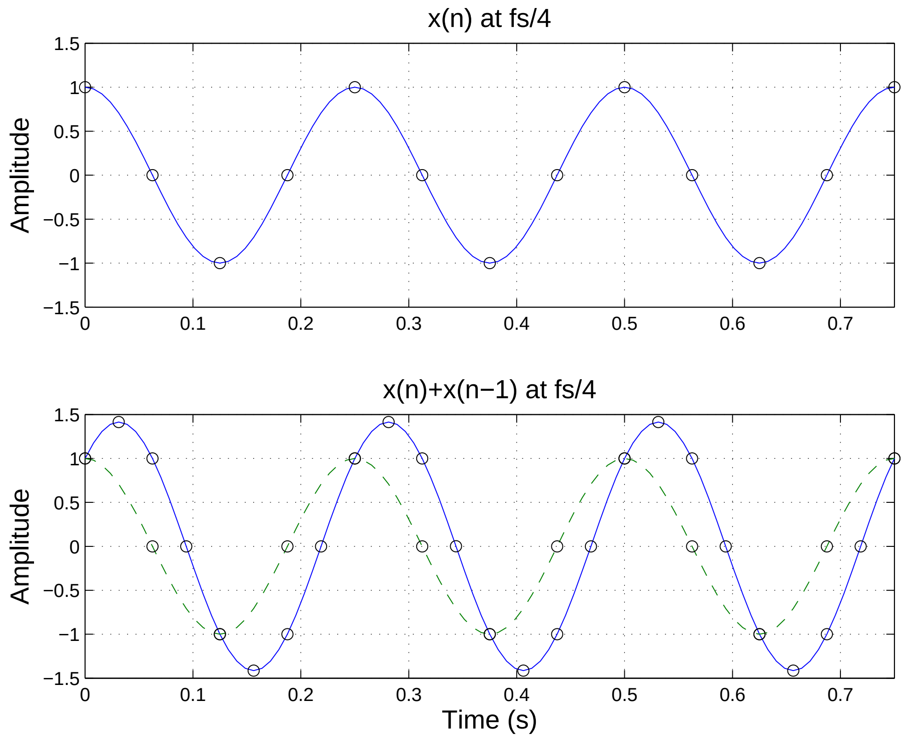


Figure 5: Filter behaviour at $f_s/4$.

- This filter is *delaying* this frequency by half a sample.
- In fact, this filter delays *all* frequencies by half a sample.

Linear Phase Filters

- Filters that delay all frequencies by the same amount are called *linear phase* filters.
- Linear phase filters have a symmetric impulse response.

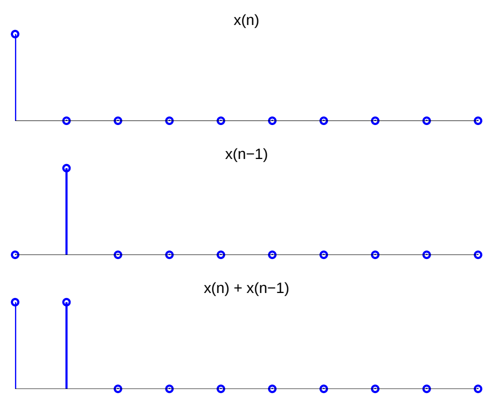


Figure 6: Filter impulse response.

- For this filter, the impulse response is symmetric about sample 0.5, which corresponds to a waveform delay of one-half sample at all frequencies.

Phase Delay

- The value about which the impulse response is symmetric is the *phase delay* of the filter.

> A "simple waveform delay" means the waveform will not change with a change in frequency.

- Linear phase is desirable because it delays all frequencies by the same number of samples and that means **no phase distortion**.

Changing Filter Coefficients

- Consider the following variation on the two-point averager (lowpass filter):

$$y(n) = x(n) - x(n-1).$$

How does changing the addition to a subtraction change the filter?

> Changing the addition to a subtraction changes the *coefficients* of the filter.

- At DC the output becomes

$$\begin{aligned} y(n) &= x_1(n) - x_1(n-1) \\ &= \quad [A, A, A, ...] \\ &\quad - [0, A, A, A, ...] \\ &= \quad [A, 0, 0, 0, ...] \end{aligned}$$

- At the Nyquist limit the output becomes

$$\begin{aligned} y(n) &= x_2(n) - x_2(n-1) \\ &= \quad [A, -A, A, ...] \\ &\quad - [0, A, -A, A, ...] \\ &= \quad [A, -2A, 2A, -2A, ...] \end{aligned}$$

Notch and Bandpass Filters

- Consider next, changing the delay value of the second term:
$$y(n) = x(n) + x(n-2).$$
- This changes the filter order to 2 and effectively sets the $x(n-1)$ term to zero.

The filter *order* is the value of its highest delay.

- This filter passes both DC and the Nyquist limit, but attenuates $f_s/4$. It is a notch filter.
- The filter given by
$$y(n) = x(n) - x(n-2)$$
rejects DC and the Nyquist limit, and boosts $f_s/4$. It is a bandpass filter.

Plots of simple filters

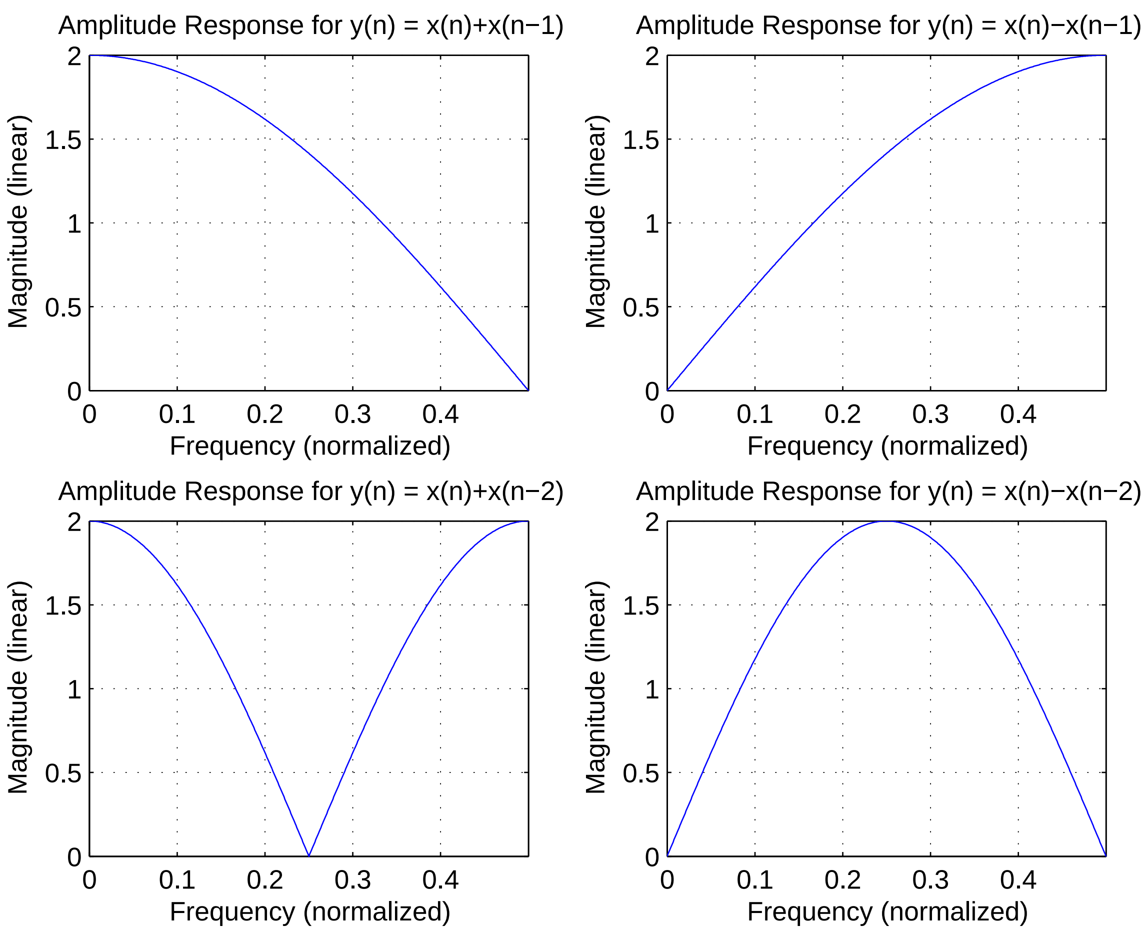


Figure 7: Amplitude Responses for simple filters

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Increasing the Filter Order

- Let's return now to the simple low-pass filter

$$y(n) = x(n) + x(n-1)$$

- Increasing the order will increase the number of samples averaged

$$y(n) = x(n) + x(n-1) + x(n-2),$$

and the waveform will be smoothed (with a more gentle slope to zero) which corresponds to a lowered cutoff frequency

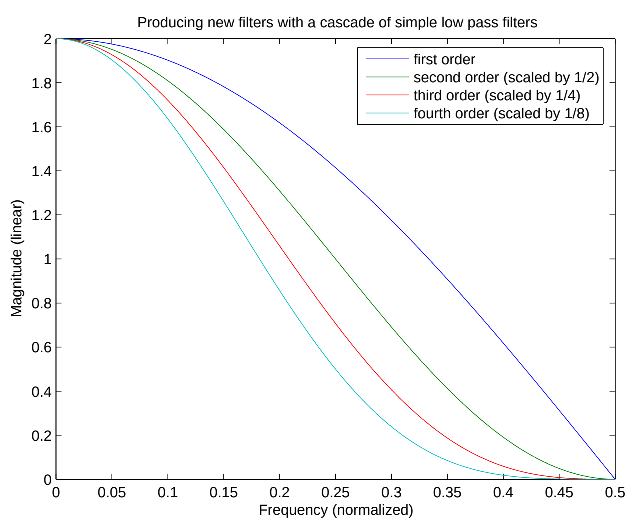


Figure 8: Lowpass filters of increasing order.

Generalized FIR filter

- Several different (nonrecursive) filters can be made by changing the delay and the coefficients of the filter terms,

$$\begin{aligned} y(n) &= b_0 x(n) + b_1 x(n-1) + \ldots \\ & \quad b_2 x(n-2) + \ldots + b_M x(n-M), \end{aligned}$$

  where M is the maximum delay and thus the order of the filter.

- A filter can be defined simply by a set of coefficients. For example if

$$b_k = \{1, 3, 3, 1\},$$

  the filter is third order (has a maximum delay of $M = 3$), and can be expanded into the difference equation

$$y(n) = x(n) + 3x(n-1) + 3x(n-2) + x(n-3)$$

Coefficients as Impulse Response

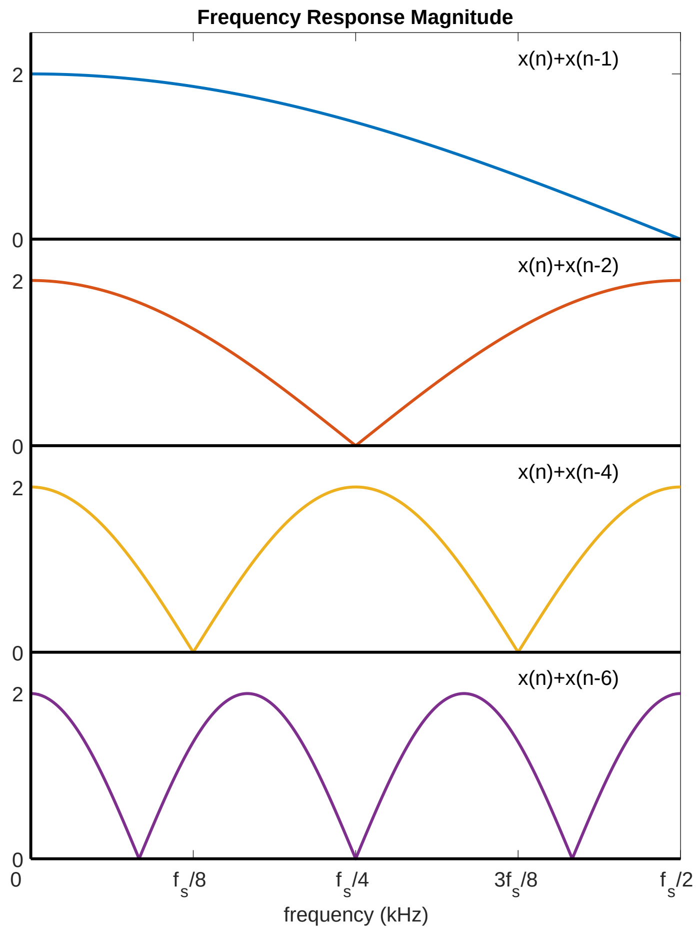
- The impulse response $y(n) = h(n)$ is equivalent to coefficients of our FIR filter b_k.
- This can be shown using the general FIR equation, with input $x(n) = \delta(n)$ (recall δ only has a nonzero value when $n = 0$).

$$
\begin{aligned}
h(0) &= b_0\delta(0) + b_1\cancelto{0}{\delta(0-1)} + b_2\cancelto{0}{\delta(0-2)} + \ldots \\
&= b_0, \\
h(1) &= b_0\cancelto{0}{\delta(1)} + b_1\delta(1-1) + b_2\cancelto{0}{\delta(1-2)} + \ldots \\
&= b_1, \\
h(2) &= b_0\delta(2) + b_1\delta(2-1) + b_2\delta(2-2) + \ldots \\
&= b_2, \\
\ldots
\end{aligned}
$$

- When the relation between the input $x(n)$ and the output $y(n)$ of the FIR filter is expressed in terms of the input and impulse response, we say the output is obtained by *convolving* the sequences $x(n)$ and $h(n)$.

Increasing the phase delay

- Again returning to the simple low pass filter...
- What happens when the delay is increased?



- Notice appearance of regularly spaced peaks/notches (like teeth of a "comb").

The Feedforward Comb Filter

- Increasing the delay of the "simple" first-order lowpass filter

$$y(n) = x(n) + gx(n - M),$$

where g is the coefficient multiplying the delay, calls for use of a delay line:

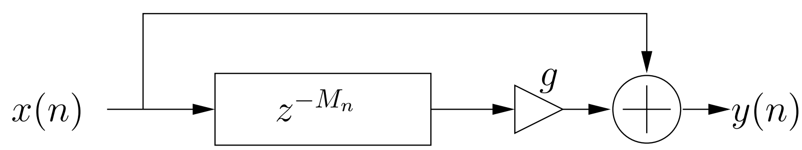


Figure 9: A feedforward comb filter.

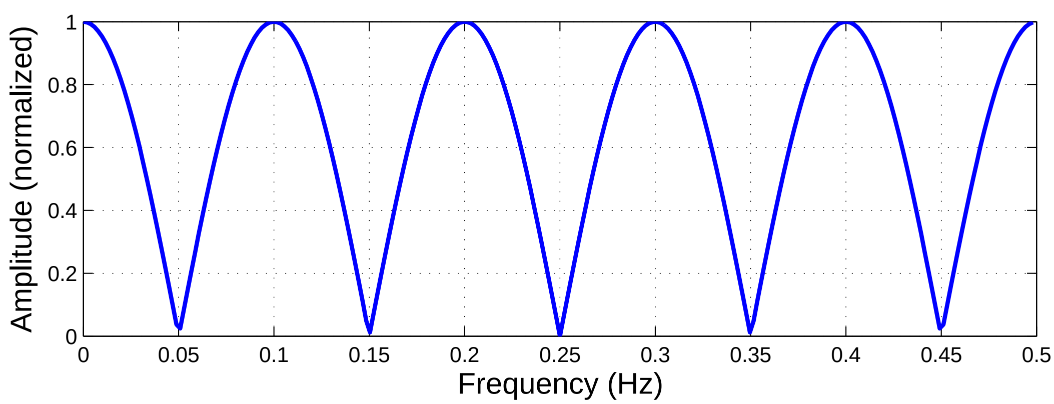


Figure 10: The comb filter magnitude response.

The Delay Line

- The **delay line** is an elementary functional unit modeling *acoustic propagation delay* (e.g. digital waveguide models and delay effects processors).

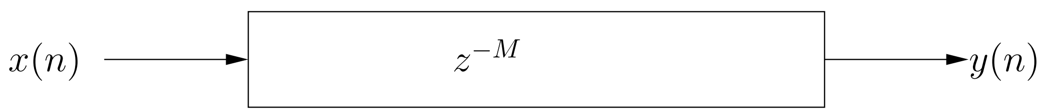


Figure 11: The M-sample delay line.

- The delay line introduces a time delay (phase delay of M samples) between input $x(n)$ and output $y(n)$:

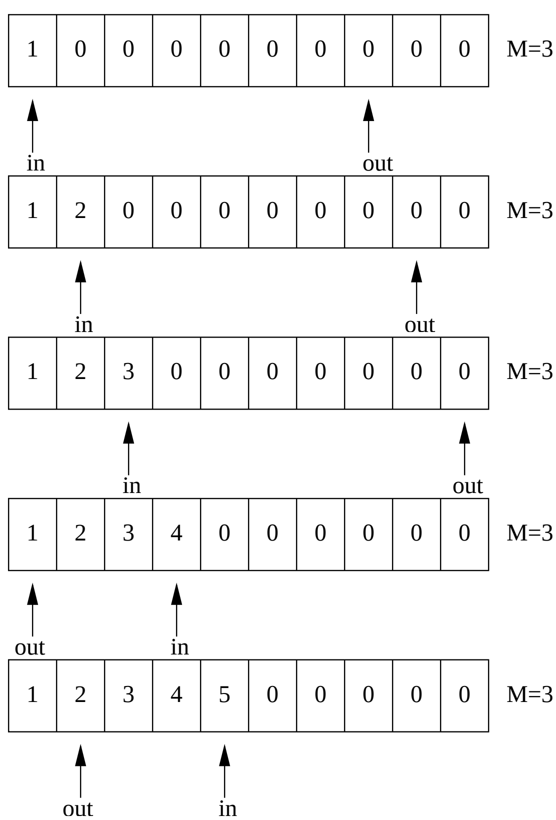
$$y(n) = x(n - M), \quad n = 0, 1, 2, \ldots$$

- It is linear phase (delay is the same at all frequencies) as can be seen by the **symmetry of the impulse response** about the M^{th} sample.

Circular Delay Line

- If M changes over time, delay may be implemented using write (in) and read (out) pointers into a larger buffer.
- The delay is set by having the read (out) pointer "chase" the write (in) pointer by the desired delay.

input sequence: 1, 2, 3, 4, 5, etc.



output sequence: 0, 0, 0, 1, 2, 3, 4, etc.

Practical Example

- If the wave propagates $L = 1$ meter:
  - Sampling rate:
$$f_s = 44100$$
  - Temporal sampling period:
$$T = \frac{1}{44100}$$
  - Spatial sampling period (distance traveled in one sample assuming propagation speed of 340 m/s):
$$X = cT = \frac{340}{44100} = 7.7\text{mm}$$
  - Number of samples delay:
$$M = \frac{L}{X} = \frac{1}{cT} = \frac{1}{0.0077} = 130$$

The Delay Line Coefficent g

- The feedforward coefficient g controls the proportion of the delay signal in the output.

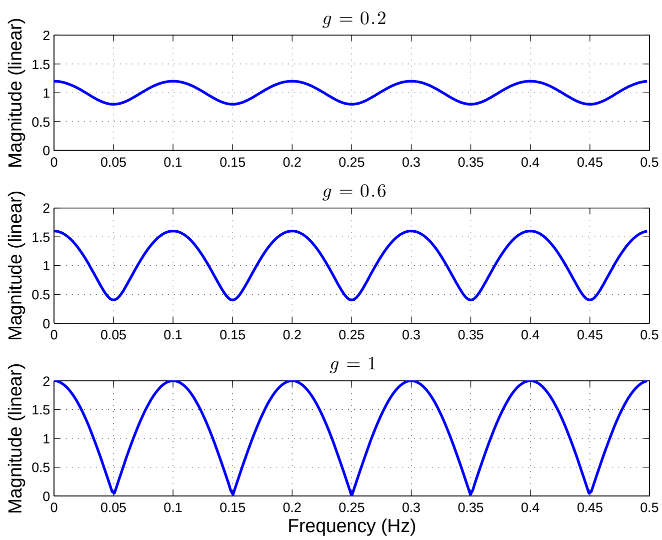


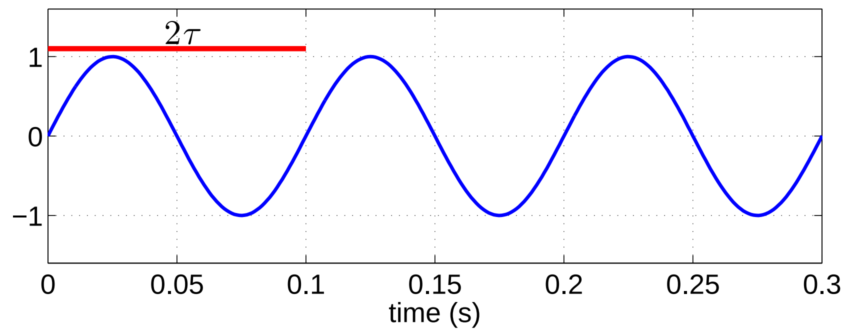
Figure 12: Delayline coefficient controlls notch attenuation and peak gain.

- It can be viewed as the DEPTH parameter, setting
  - the amount of **gain** at the maxima,
  - the amount of **attenuation** at the minima,

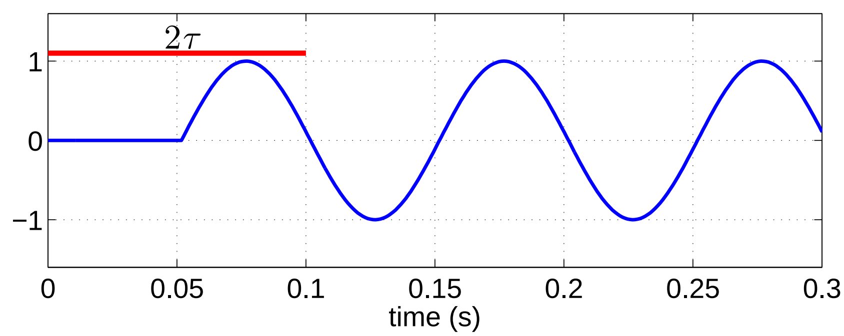
  that is, the *depth* from the peaks to the notches.
- Has a range from 0 to 1 (1 corresponds to maximum depth).

Cancellation at Notch Frequencies

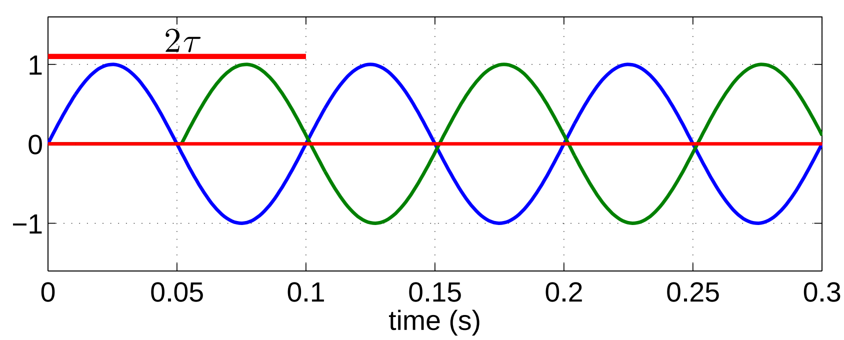
- Consider a sinusoid at $f = 1/(2\tau)$ (period of 2τ):



- Delaying that sinusoid by τ (1/2 a period) yields:

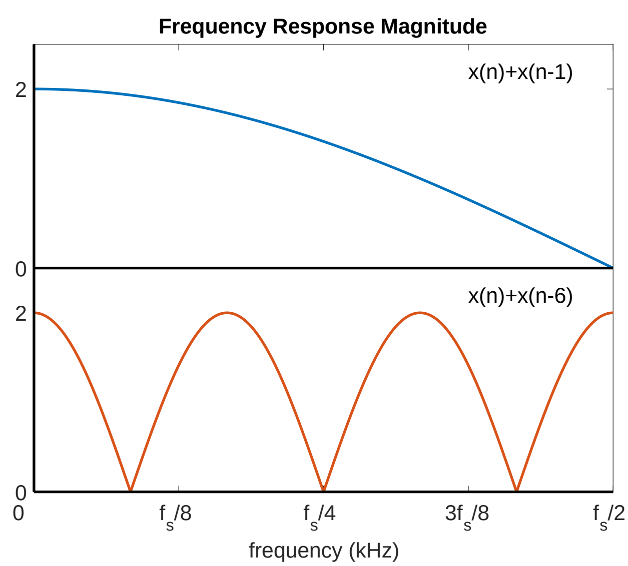


- Summing with original yields complete cancellation:



Relating τ to delay of M samples

- For $y(n) = x(n) + x(n - M)$ delay is M samples or
$$\tau = \frac{M}{fs} \text{ seconds.}$$
- There is complete attenuation (notch) at frequency
$$f = \frac{1}{2\tau} = \frac{1}{2M/f_s} = \frac{f_s}{2M}$$
and at odd harmonics $3f, 5f, \ldots$ (up to Nyquist limit).
- For $M = 1$ (lowpass) there is 1 notch at $f_s/2$.



- For $M = 6$ there are notches at $f_s/12, f_s/4, 5f_s/12$.

Why Spectral Notches?

- Notches occur in the spectrum as a result of **destructive interference**.
- Recall that delaying a sine tone 180 degrees (1/2 a cycle) and summing with the original will cause the signal to disappear at the output.

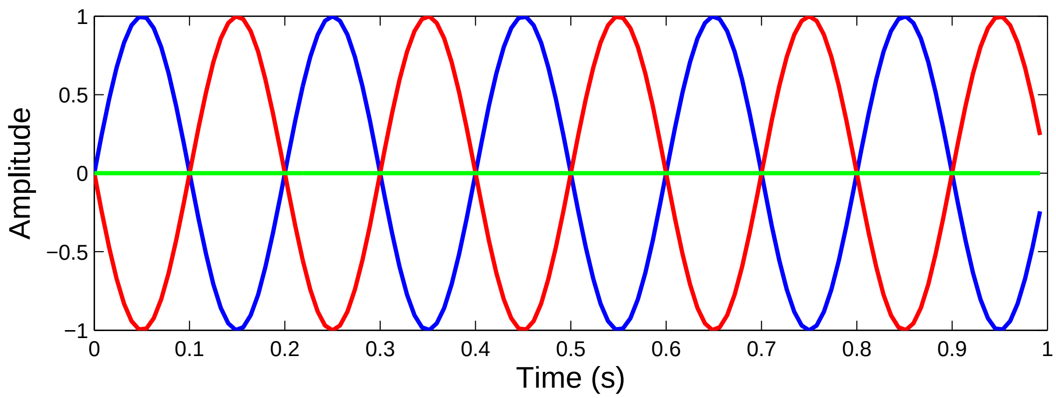


Figure 13: Complete destructive interference.

- If a sinusoid has a frequency of f_0, then a delay of 1/2 cycle corresponds to a delay of

$$\tau = \frac{1}{2f_0} \text{ seconds}$$

or

$$M = \frac{f_s}{2f_0} \text{ samples.}$$

Delay Parameter M

- A delay of $M = f_s/(2f_0)$ samples in the comb filter will yield a *notch* (complete cancelation) at
$$f_0 = \frac{f_s}{2M} \text{ Hz},$$
as well as notches at **odd harmonics** of f_0.

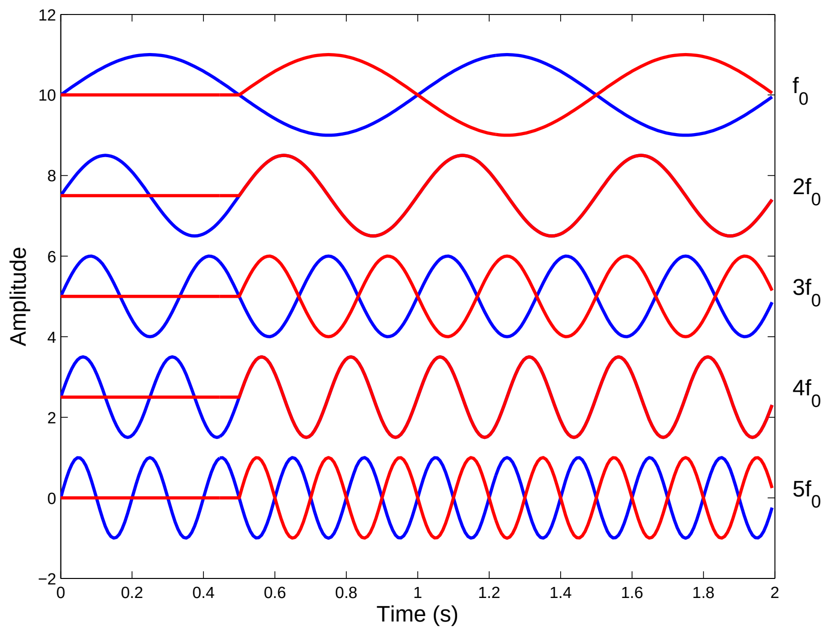


Figure 14: Destructive interference occurs at odd harmonics of the fundamental frequency.

- The well known *flanger* is a feedforward comb filter with a time-varying delay $M(n)$ (see flanging.mov).