Music 270a: Modulation

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Additive Synthesis Caveat

- **Drawback**: it often requires many oscillators to produce good quality sounds, and can be very computationally demanding.

- Also, many functions are useful only for a limited range of pitch and loudness. For example,

  - the timbre of a piano played at A4 is different from one played at A2;

  - the timbre of a trumpet played loudly is quite different from one played softly at the same pitch.

- It is possible however, to use some knowledge of acoustics to determine functions:

  - e.g., in specifying amplitude envelopes for each oscillator, it is useful to know that in many acoustic instruments the higher harmonics attack last and decay first.

Harmonics and Pitch

- Waveforms that contain more than one frequency component (i.e. aren't purely sinusoidal) may still be periodic.

- If frequency components are (more or less) **integer multiples** of some fundamental frequency f_0, producing a spectrum with (more or less) evenly spaced frequency components, they are called *harmonics*.

- Signals with harmonic spectra have a periodic waveform where the period is the inverse of the fundamental frequency (often the *sounding* frequency).

- *Pitch* is our subjective response to a fundamental frequency within the audio range.

- The harmonics contribute to the *timbre* or *quality* of a sound, but do not necessarily alter the *pitch*.

Beat Notes

- What happens when two sinusoids **that are not harmonically related** are added?
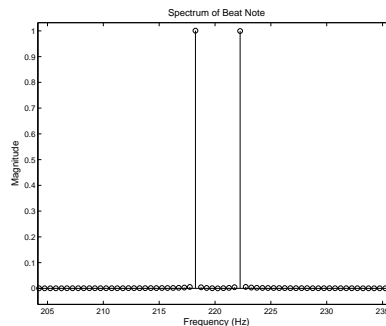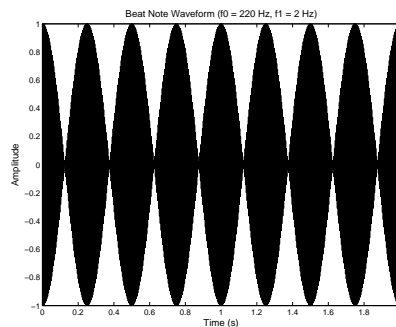

Figure 1: Beat Note made by adding sinusoids at frequencies 218 Hz and 222 Hz.

- The resulting waveform shows a periodic, low frequency amplitude envelope superimposed on a higher frequency sinusoid.

- The **beat note** comes about by adding two sinusoids that are very close in frequency.

Multiplication of Sinusoids

- What happens when we multiply a low frequency sinusoids with a higher frequency sinusoid (or any two sinusoids)?

$$\sin(2\pi(220)t)\cos(2\pi(2)t)$$
$$= \left(\frac{e^{j2\pi(220)t} - e^{-j2\pi(220)t}}{2j}\right)\left(\frac{e^{j2\pi(2)t} + e^{-j2\pi(2)t}}{2}\right)$$
$$= \frac{1}{4j}\left[e^{j2\pi(222)t} - e^{-j2\pi(222)t} + e^{j2\pi(218)t} - e^{-j2\pi(218)t}\right]$$
$$= \frac{1}{2}\left[\sin(2\pi(222)t) + \sin(2\pi(218)t)\right]$$

- The **product** of two real sinusoids results in the **sum** of two real sinusoids (having different frequencies).

- Interestingly, the resulting spectral components (those in the sum) are not at the frequencies in the product. Rather, they are at their **sum** and the **difference**.

- Sinusoidal multiplication can therefore be expressed as an addition.

Amplitude Modulation

- Modulation is the alteration of the amplitude, phase, or frequency of an oscillator in accordance with another signal.

- The oscillator being modulated is the *carrier*, and the altering signal is called the *modulator*.

- Amplitude modulation, therefore, is the alteration of the amplitude of a carrier by a modulator.

- The spectral components generated by a modulated signal are called *sidebands*.

- There are three main techniques of amplitude modulation:

    - Ring modulation
    - "Classical" amplitude modulation
    - Single-sideband modulation

Ring Modulation

- Ring modulation (RM), introduced as the beat note waveform, occurs when modulation is applied directly to the amplitude input of the carrier modulator:

$$x(t) = \cos(2\pi f_\Delta t) \cos(2\pi f_c t).$$

- Recall that this multiplication can also be expressed as the sum of sinusoids using the inverse of Euler's formula:

$$x(t) = \frac{1}{2} \cos(2\pi f_1 t) + \frac{1}{2} \cos(2\pi f_2 t),$$
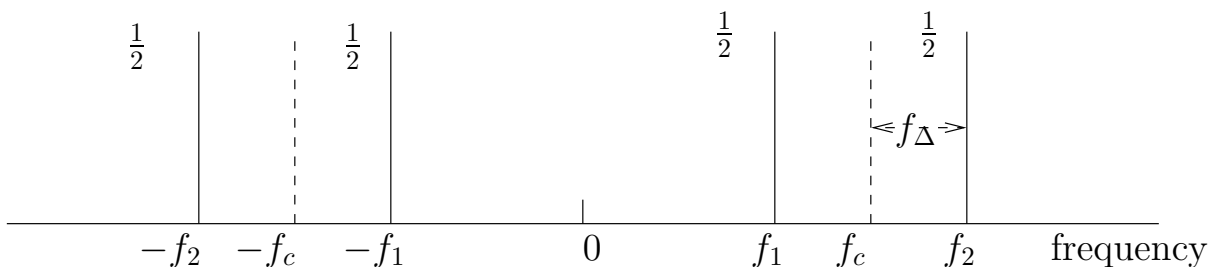
where $f_1 = f_c - f_\Delta$ and $f_2 = f_c + f_\Delta$.

Figure 2: Spectrum of ring modulation.

Ring Modulation cont.

- Notice again that neither the carrier frequency nor the modulation frequency are present in the spectrum.
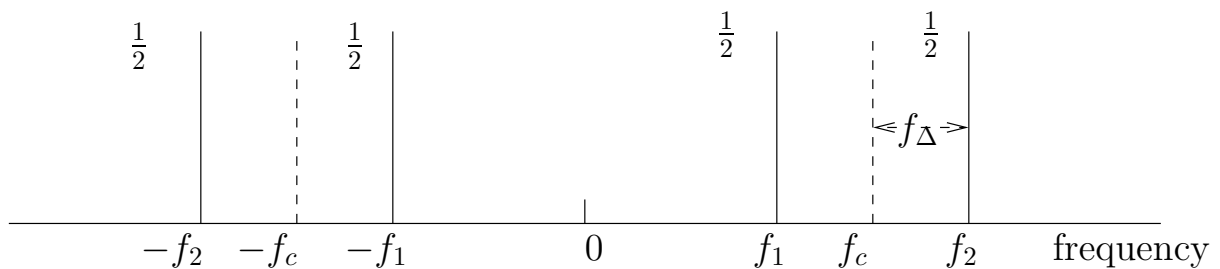

Figure 3: Spectrum of ring modulation.

- Because of its spectrum, RM is also sometimes called *double-sideband (DSB)* modulation.

- Ring modulating can be realized without oscillators just by multiplying two signals together.

- The multiplication of two complex sounds produces a spectrum containing frequencies that are the sum and difference between each of the frequencies present in each of the sounds.

- The number of components in RM is *two times the number of frequency components in one signal multiplied by the number of frequency components in the other*.

"Classic" Amplitude Modulation

- "Classic" amplitude modulation (AM) is the more general of the two techniques.

- In AM, the modulating signal includes a constant, a DC component, in the modulating term,

$$x(t) = [A_0 + \cos(2\pi f_\Delta t)] \cos(2\pi f_c t).$$

- Multiplying out the above equation yields

$$x(t) = A_0 \cos(2\pi f_c t) + \cos(2\pi f_\Delta t) \cos(2\pi f_c t).$$

- The first term in the result above shows that **the carrier frequency is actually present in the resulting spectrum**.

- The second term can be expanded in the same way as was done for ring modulation, using the inverse Euler formula (left as an exercise).

RM and AM Spectra

- Where the centre frequency f_c was absent in RM, it is present in classic AM. The sidebands are identical.
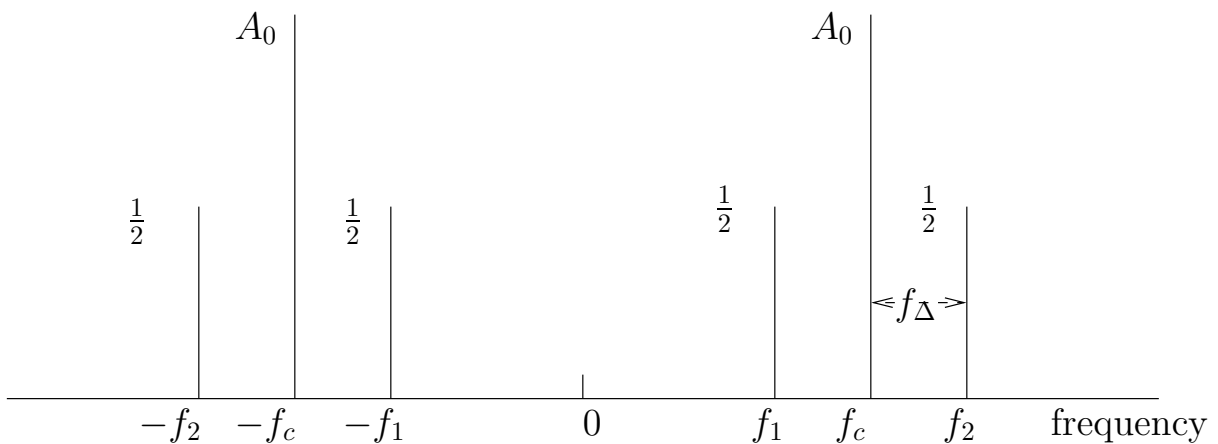
$$A_0 \quad\quad\quad\quad\quad A_0$$

$$\frac{1}{2} \quad\quad \frac{1}{2} \quad\quad\quad \frac{1}{2} \quad\quad \frac{1}{2}$$

$$\overset{\ll f_\Delta \gg}{}$$

$$-f_2 \quad -f_c \quad -f_1 \quad\quad 0 \quad\quad f_1 \quad f_c \quad f_2 \quad \text{frequency}$$

Figure 4: Spectrum of amplitude modulation.

$$\frac{1}{2} \quad\quad \frac{1}{2} \quad\quad\quad \frac{1}{2} \quad\quad \frac{1}{2}$$

$$\overset{\ll f_\Delta \gg}{}$$

$$-f_2 \quad -f_c \quad -f_1 \quad\quad 0 \quad\quad f_1 \quad f_c \quad f_2 \quad \text{frequency}$$
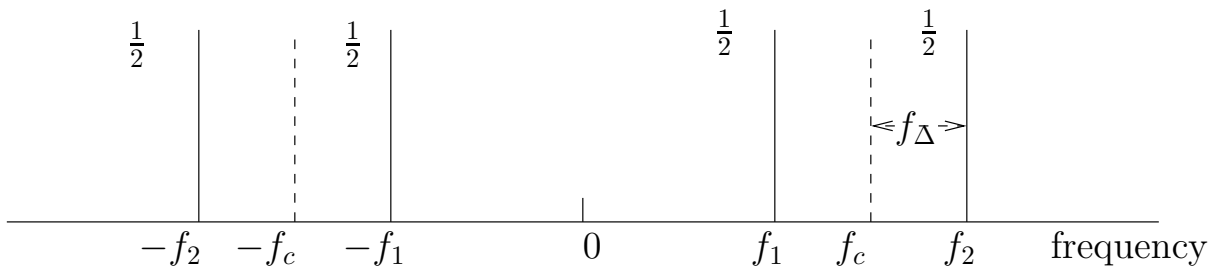
Figure 5: Spectrum of ring modulation.

- A DC offset A_0 in the modulating term therefore has the effect of including the centre frequency f_c at an amplitude equal to the offset.

RM and AM waveforms

- Because of the DC component, the modulating signal is often *unipolar*—the entire signal is above zero and the instantaneous amplitude is always positive[1].
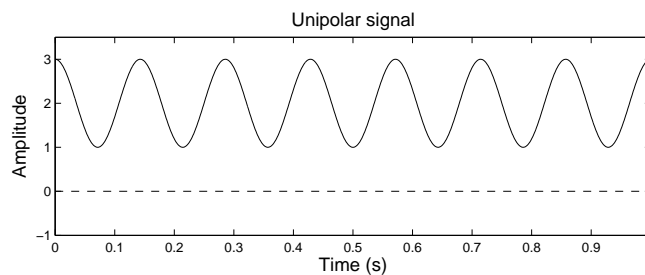

Figure 6: A unipolar signal.

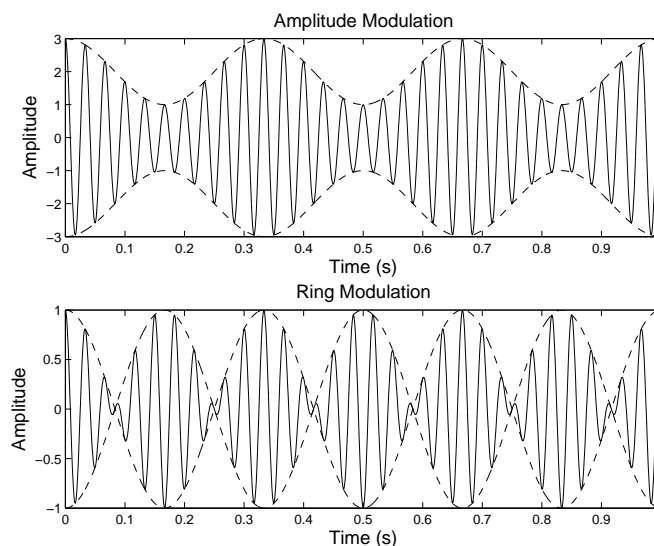- The effect of a DC offset in the modulating term is seen in the difference between AM and RM signals.

Figure 7: AM (top) and RM (bottom) waveforms.

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[1]The signal without the DC offset that oscillates between the positive and negative is called *bipolar*

Single-Sideband modulation

- Single-sideband modulation changes the amplitudes of two carrier waves that have quadrature phase relationship.

- Single-sideband modulation is given by

$$x_{ssb}(t) = x(t) \cdot \cos(2\pi f_c t) - \mathcal{H}\{x(t)\} \cdot \sin(2\pi f_c t),$$

where $\mathcal{H}$ is the Hilbert transform.

Frequency Modulation (FM)

- Where AM is the modulation of amplitude, frequency modulation (FM) is the modulation of the carrier *frequency* with a modulating signal.

- Untill now we've seen signals that do not change in frequency over time—how do we modify the signal to obtain a time-varying frequency?

- One approach might be to concatenate small sequences, with each having a given frequency for the duration of the unit.
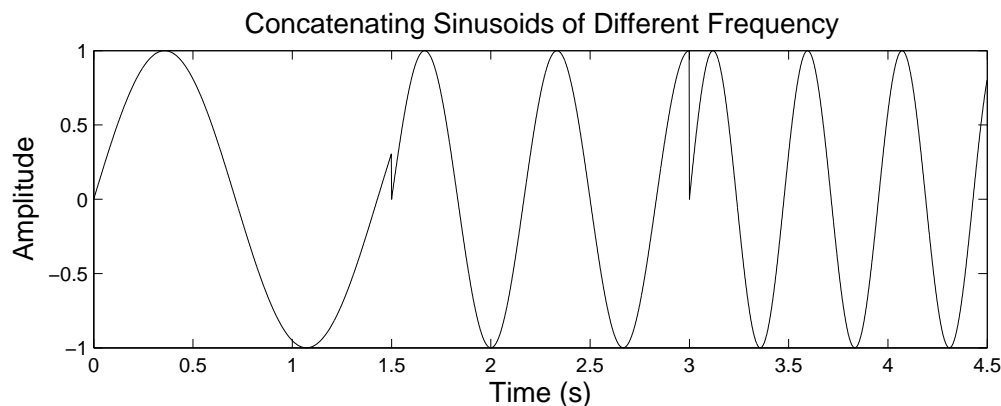

Figure 8: A signal made by concatenating sinusoids of different frequencies will result in discontinuities if care is not taken to match the initial phase.

- Care must be taken in not introducing discontinuities when lining up the phase of each segment.

Chirp Signal

- A **chirp** signal is one that sweeps linearly from a low to a high frequency.

- To produce a chirping sinusoid, modifying its equation so the frequency is time-varying will likely produce better results than concatenating segments.

- Recall that the original equation for a sinusoid is given by
$$x(t) = A\cos(\omega_0 t + \phi)$$
where the instantaneous phase, given by $(\omega_0 t + \phi)$, changes *linearly with time*.

- Notice that the time derivative of the phase is the radian frequency of the sinusoid ω_0, which in this case is a constant.

- More generally, if
$$x(t) = A\cos(\theta(t)),$$
the instantaneous frequency is given by
$$\omega(t) = \frac{d}{dt}\theta(t).$$

Chirp Sinusoid

- Now, let's make the phase *quadratic*, and thus *non-linear with respect to time*.

$$\theta(t) = 2\pi\mu t^2 + 2\pi f_0 t + \phi.$$

- The instantaneous frequency (the derivative of the phase θ), becomes

$$\omega_i(t) = \frac{d}{dt}\theta(t) = 4\pi\mu t + 2\pi f_0,$$

which in Hz becomes

$$f_i(t) = 2\mu t + f_0.$$

- Notice the frequency is no longer constant but changing linearly in time.

- To create a sinusoid with frequency sweeping linearly from f_1 to f_2, consider the equation for a line $y = mx + b$ to obtain

$$f(t) = \frac{f_2 - f_1}{2T}t + f_1,$$

where T is the duration of the sweep.

Sweeping Frequency

- The time-varying expression for the instantaneous frequency can be used in the original equation for a sinusoid,

$$x(t) = A\cos(2\pi f(t)t + \phi).$$
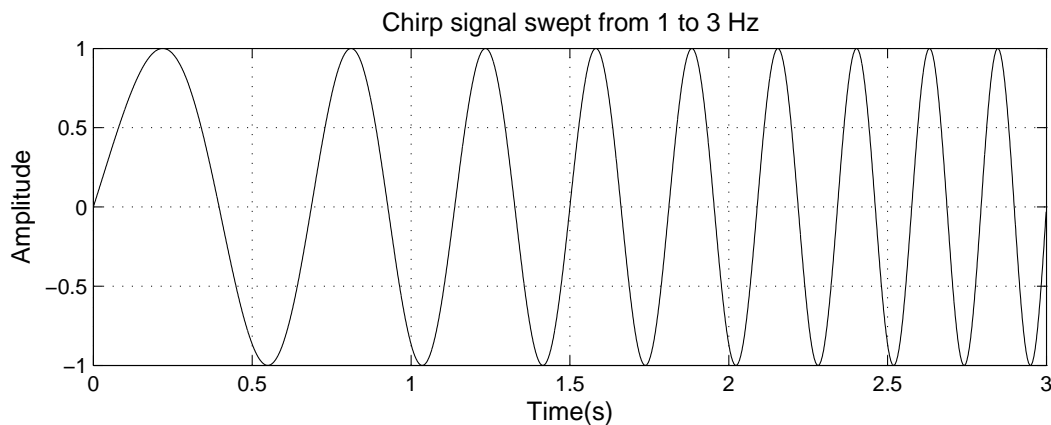
Chirp signal swept from 1 to 3 Hz

Figure 9: A chirp signal from 1 to 3 Hz.

- Phase summary:

  - If the instantaneous phase $\theta(t)$ is constant, the frequency is zero (DC).
  - If $\theta(t)$ is linear, the frequency is fixed (constant).
  - if $\theta(t)$ is quadratic, the frequency changes linearly with time.

Vibrato simulation

- Vibrato is a term used to describe a wavering of pitch.

- Vibrato (in varying amounts) occurs very naturally in the singing voice and in many "sustained" instruments (where the musician has control after the note has been played), such as the violin, wind instruments, the theremin, etc.).

- In Vibrato, the frequency does not change linearly but rather *sinusoidally*, creating a sense of a wavering pitch.

- Since the instantaneous frequency of the sinusoid is the derivative of the instantaneous phase, and the derivative of a sinusoid is a sinusoid, a vibrato can be simulated by applying a sinusoid to the instantaneous phase of a carrier signal:

$$x(t) = A_c \cos(2\pi f_c t + A_m \cos(2\pi f_m t + \phi_m) + \phi_c),$$

that is, by FM synthesis.

FM Vibrato

- FM synthesis can create a vibrato effect, where the instantaneous frequency of the carrier oscillator varies over time according to the parameters controlling

  - the *width* of the vibrato (the deviation from the carrier frequency)
  - the *rate* of the vibrato.

- The *width* of the vibrato is determined by the amplitude of the modulating signal, A_m.

- The *rate* of vibrato is determined by the frequency of the modulating signal, f_m.

- In order for the effect to be perceived as vibrato, the vibrato rate **must be below the audible frequency range** and the width made quite small.

FM Synthesis of Musical Instruments

- When the vibrato rate is in the audio frequency range, and when the width is made larger, the technique can be used to create a broad range of distinctive timbres.

- Frequency modulation (FM) synthesis was invented by John Chowning at Stanford University's *Center for Computer Research in Music and Acoustics* (CCRMA).

- FM synthesis uses fewer oscillators than either additive or AM synthesis to introduce more frequency components in the spectrum.

- Where AM synthesis uses a signal to modulate the *amplitude* of a carrier oscillator, FM synthesis uses a signal to modulate the *frequency* of a carrier oscillator.

Frequency Modulation

- The general equation for an FM sound synthesizer is given by

$$x(t) = A(t) \left[\cos(2\pi f_c t + I(t) \cos(2\pi f_m t + \phi_m) + \phi_c \right],$$

where

$$
\begin{aligned}
A(t) &\triangleq \ \text{the time varying amplitude} \\
f_c &\triangleq \ \text{the carrier frequency} \\
I(t) &\triangleq \ \text{the modulation index} \\
f_m &\triangleq \ \text{the modulating frequency} \\
\phi_m, \phi_c &\triangleq \ \text{arbitrary phase constants.}
\end{aligned}
$$

Modulation Index

- The function $I(t)$, called the *modulation index envelope*, determines significantly the harmonic content of the sound.

- Given the general FM equation

$$x(t) = A(t)\left[\cos(2\pi f_c t + I(t)\cos(2\pi f_m t + \phi_m) + \phi_c\right],$$

the instantaneous frequency $f_i(t)$ (in Hz) is given by

$$
\begin{aligned}
f_i(t) &= \frac{1}{2\pi}\frac{d}{dt}\theta(t) \\
&= \frac{1}{2\pi}\frac{d}{dt}\left[2\pi f_c t + I(t)\cos(2\pi f_m t + \phi_m) + \phi_c\right] \\
&= \frac{1}{2\pi}[2\pi f_c - I(t)\sin(2\pi f_m t + \phi_m)2\pi f_m + \\
&\qquad\qquad \frac{d}{dt}I(t)\cos(2\pi f_m t + \phi_m)] \\
&= f_c - I(t)f_m\sin(2\pi f_m t + \phi_m) + \\
&\qquad\qquad \frac{1}{2\pi}\frac{d}{dt}I(t)\cos(2\pi f_m t + \phi_m).
\end{aligned}
$$

Modulation Index cont.

- Without going any further in solving this equation, it is possible to get a sense of the effect of the modulation index $I(t)$ from

$$f_i(t) = f_c - I(t)f_m \sin(2\pi f_m t + \phi_m) + \frac{dI(t)}{dt}\cos(2\pi f_m t + \phi_m)/2\pi.$$

- We may see that if $I(t)$ is a constant (and it's derivative is zero), the third term goes away and the instantaneous frequency becomes

$$f_i(t) = f_c - I(t)f_m \sin(2\pi f_m t + \phi_m).$$

- Notice now that in the second term, the quantity $I(t)f_m$ multiplies a sinusoidal variation of frequency f_m, indicating that $I(t)$ **determines the maximum amount by which the instantaneous frequency deviates from the carrier frequency** f_c.

- Since the modulating frequency f_m is at audio rates, this translates to addition of harmonic content.

- Since $I(t)$ is a function of time, the harmonic content, and thus the timbre, of the synthesized sound may vary with time.

FM Sidebands

- The upper and lower sidebands produced by FM are given by $f_c \pm k f_m$.
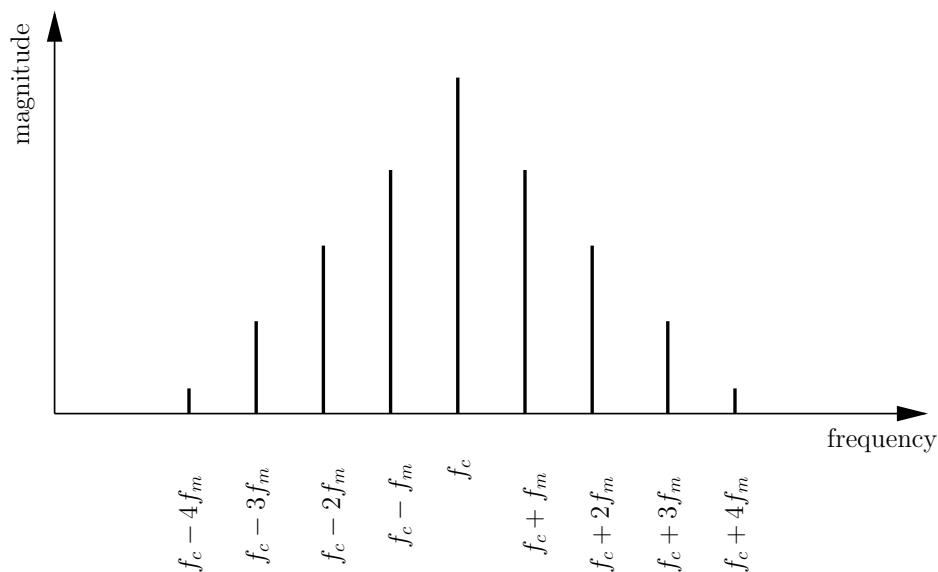

Figure 10: Sidebands produced by FM synthesis.

- The modulation index may be see as

$$I = \frac{d}{f_m},$$

where d is the amount of frequency deviation from f_c.

- When $d = 0$, the index I is also zero, and no modulation occurs. Increasing d causes the sidebands to acquire more power at the expense of the power in the carrier frequency.

Bessel Functions of the First Kind

- The amplitude of the kth sideband is given by $J_k(I)$, a Bessel function[2] of the first kind, of order k.
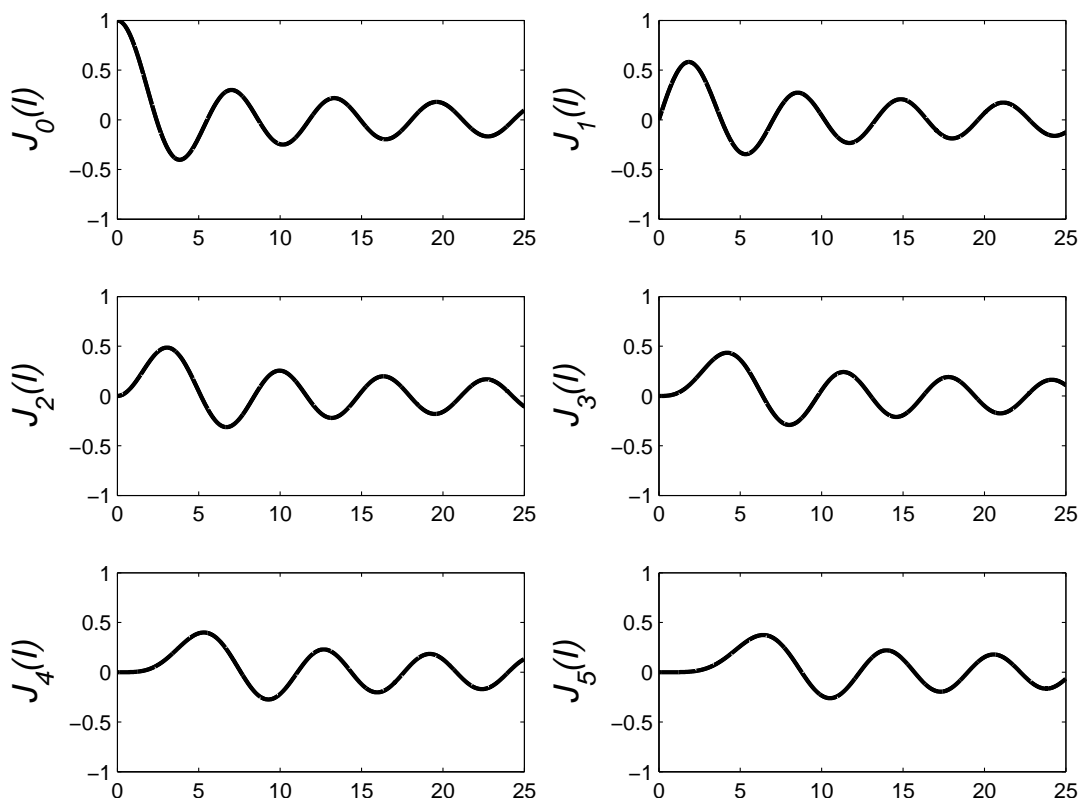

Figure 11: Bessel functions of the first kind, plotted for orders 0 through 5.

- Notice that higher order Bessel functions, and thus higher order sidebands, do not have significant amplitude when I is small.

[2]Bessel functions are solutions to Bessel's differential equation.

- Higher values of I therefore produce higher order sidebands.

- In general, the highest-ordered sideband that has significant amplitude is given by the approximate expression $k = I + 1$.

Odd-Numbered Lower Sidebands

- The amplitude of the odd-numbered lower sidebands is the appropriate Bessel function multipled by -1, since odd-ordered Bessel functions are **odd functions**. That is
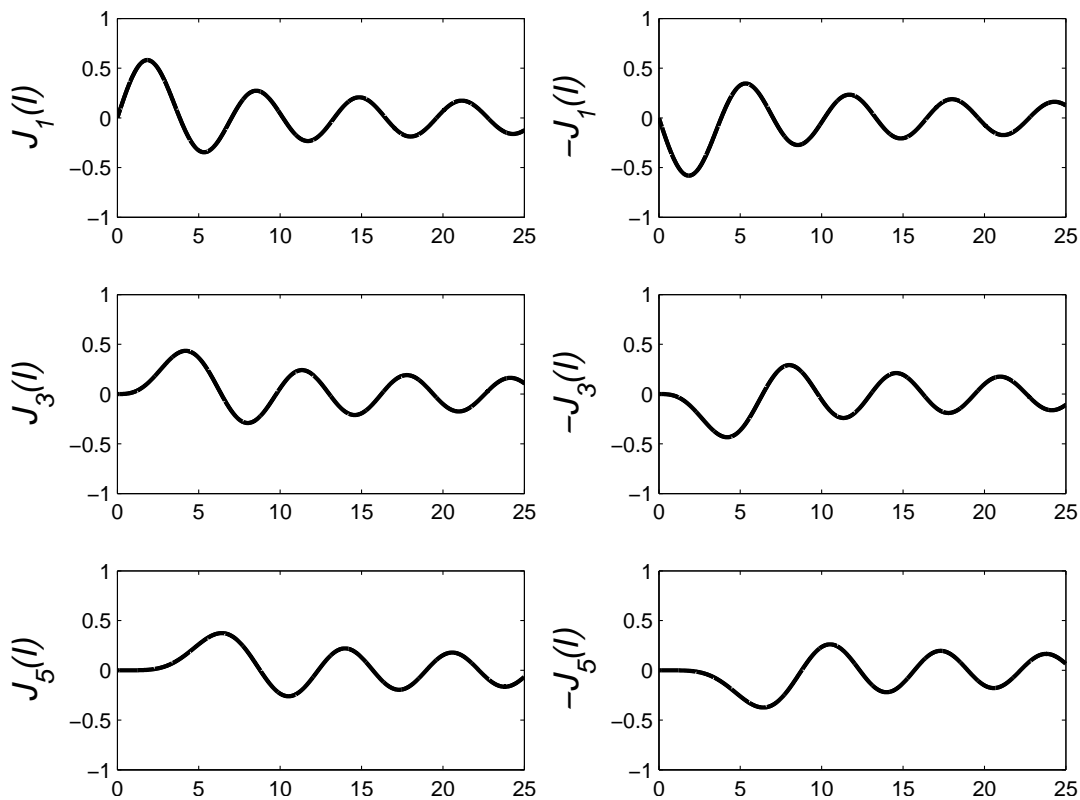
$$J_{-k}(I) = -J_k(I).$$

Figure 12: Bessel functions of the first kind, plotted for odd orders.

Effect of Phase in FM

- The phase of a spectral component does not have an audible effect unless other spectral components of the same frequency are present, i.e., there is *interference*.

- In the case of frequency overlap, the amplitudes will either *add* or *subtract*, resulting in a perceived change in the sound.

- If the FM spectrum contains frequency components below 0 Hz, they are *folded over* the 0 Hz axis to their corresponding positive frequencies (akin to *aliasing* or *folding over* the Nyquist limit).

FM Spectrum

- The frequencies present in an FM spectrum are $f_c \pm k f_m$, where k is an integer greater than zero (the carrier frequency component is at $k = 0$).
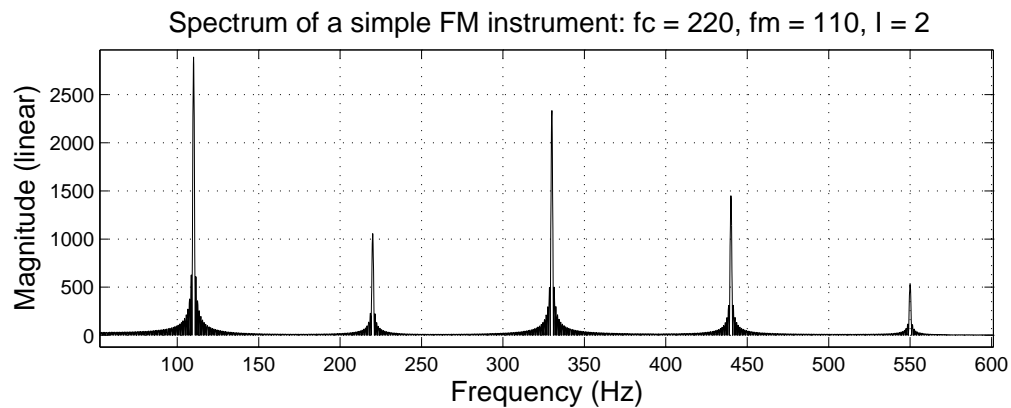

Figure 13: Spectrum of a simple FM instrument, where $f_c = 220$, $f_m = 110$, and $I = 2$.
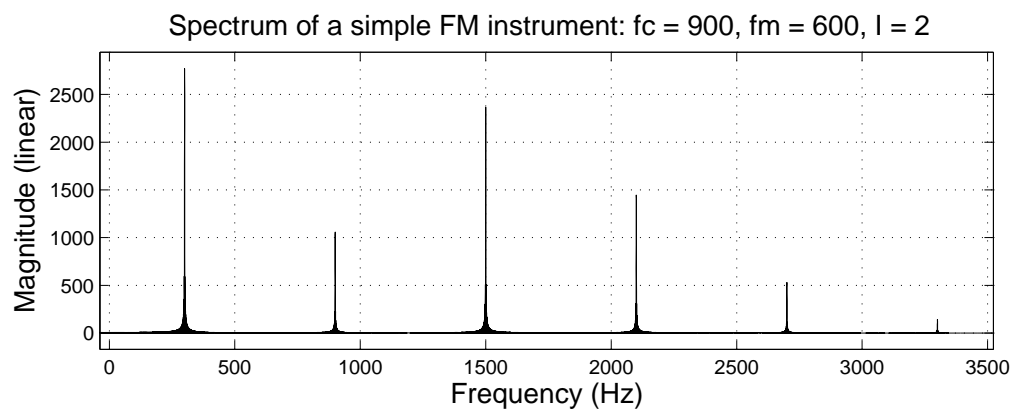

Figure 14: Spectrum of a simple FM instrument, where $f_c = 900$, $f_m = 600$, and $I = 2$.

Fundamental Frequency in FM

- In determining the fundamental frequency of an FM sound, first represent the ratio of the carrier and modulator frequencies as a reduced fraction,
$$\frac{f_c}{f_m} = \frac{N_1}{N_2},$$
where N_1 and N_2 are integers with no common factors.

- The fundamental frequency is then given by
$$f_0 = \frac{f_c}{N_1} = \frac{f_m}{N_2}.$$

- Example: a carrier frequency $f_c = 220$ and modulator frequency $f_m = 110$ yields the ratio of
$$\frac{f_c}{f_m} = \frac{220}{110} = \frac{2}{1} = \frac{N_1}{N_2}.$$
and a fundamental frequency of
$$f_0 = \frac{220}{2} = \frac{110}{1} = 110.$$

- Likewise the ratio of $f_c = 900$ to $f_m = 600$ is 3:2 and the fundamental frequency is given by
$$f_0 = \frac{900}{3} = \frac{600}{2} = 300.$$

Missing Harmonics in FM Spectrum

- For $N_2 > 1$, every N_2^{th} harmonic of f_0 is missing in the spectrum.

- This can be seen in the plot below where the ratio of the carrier to the modulator is 4:3 and $N_2 = 3$.

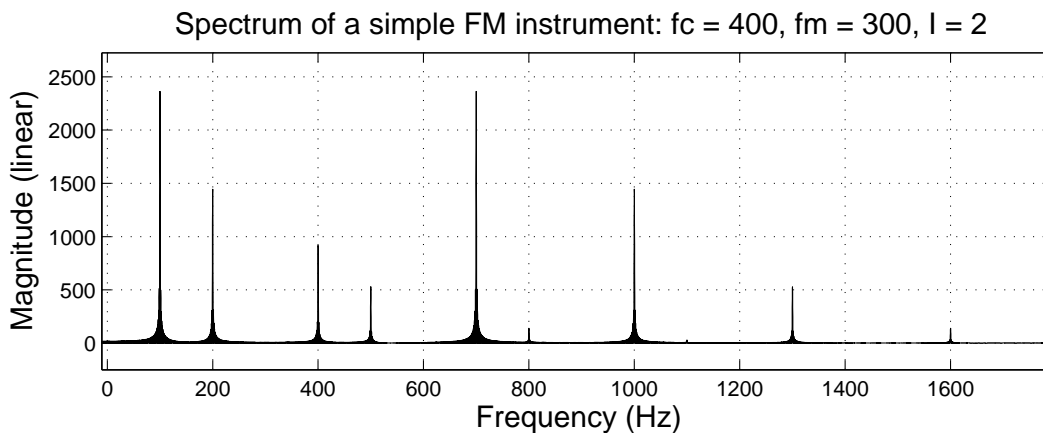Spectrum of a simple FM instrument: fc = 400, fm = 300, I = 2

Figure 15: Spectrum of a simple FM instrument, where $f_c = 400$, $f_m = 300$, and $I = 2$.

- Notice the fundamental frequency f_0 is 100 and every third multiple of f_0 is missing from the spectrum.

Some FM instrument examples

- When implementing simple FM instruments, we have several basic parameters that will effect the overall sound:

  1. The duration,

  2. The carrier and modulating frequencies

  3. The maximum (and in some cases minimum) modulating index scalar

  4. The envelopes that define how the amplitude and modulating index evolve over time.

- Using the information taken from John Chowning's article on FM (details of which appear in the text *Computer Music* (pp. 125-127)), we may develop envelopes for the following simple FM instruments:

  - bell-like tones,

  - wood-drum

  - brass-like tones
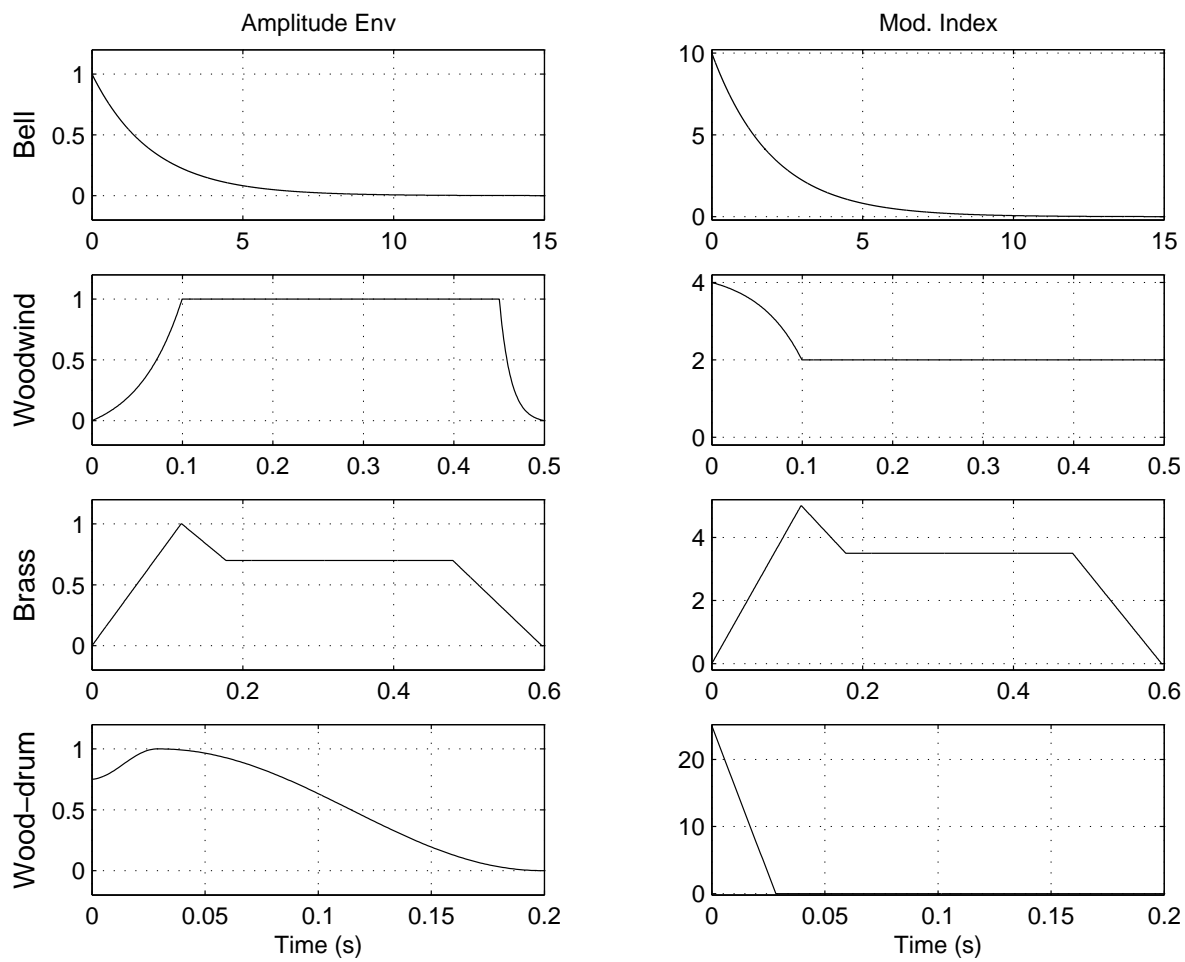
  - clarinet-like tones

Figure 16: Envelopes for FM bell-like tones, wood-drum tones, brass-like tones and clarinet tones.

Formants

- Another characteristic of sound, in addition to its spectrum, is the presence of formants, or *peaks in the spectral envelope*.

- The formants describe certain regions in the spectrum where there are strong resonances (where the amplitude of the spectral components is considerably higher).

- As an example, pronounce aloud the vowels "a", "e", "i", "o", "u" while keeping the same pitch for each. Since the pitch is the same, we know the integer relationship of the spectral components is the same.

- The formants are what allows us to hear a difference between the vowel sounds.

Formants with Two Carrier Oscillators

- In FM synthesis, the peaks in the spectral envelop can be controlled using an additional carrier oscillator.

- In the case of a single oscillator, the spectrum is centered around a carrier frequency.

- With an additional oscillator, an additional spectrum may be generated that is centered around a formant frequency.

- When the two signals are added, their spectra are combined.

- If the same oscillator is used to modulate both carriers (though likely using seperate modulation indeces), and the formant frequency is an integer multiple of the fundamental, the spectra of both carriers will combine in such a way the the components will overlap, and a peak will be created at the formant frequency.
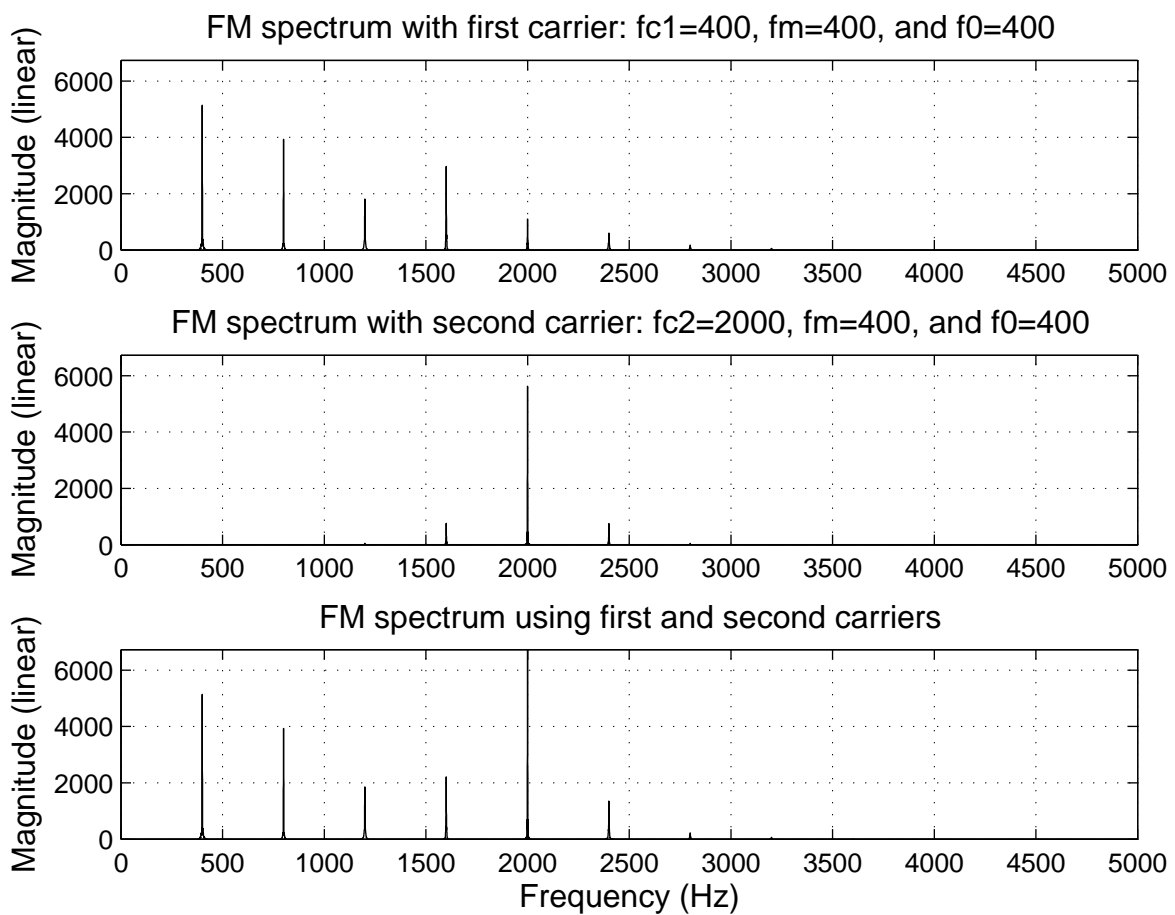
Figure 17: The spectrum of individiual and combined FM signals.

Towards a Chowning FM Trumpet

- In Figure 17, two carriers are modulated by the same oscillator with a frequency f_m.

- The index of modulation for the first and second carrier is given by I_1 and I_2/I_1 respectively.

- The value I_2 is usually less than I_1, so that the ratio I_2/I_1 is small and the spectrum does not spread too far beyond the region of the formant.

- The frequency of the second carrier f_{c2} is chosen to be a harmonic of the fundamental frequency f_0, so that it is close to the desired formant frequency f_f,

$$f_{c2} = n f_0 = \text{int}(f_f/f_0 + 0.5)f_0.$$

- This ensures that the second carrier frequency remains harmonically related to f_0.

- If f_0 changes, the scond carrier frequency will remain as close as possible to the desired formant frequency f_f while remaining an integer multiple of the fundamental frequency f_0.

Two Modulating Oscillators

- Just as the number of carriers can be increased, so can the number of modulating oscillators.

- To create even more spectral variety, the modulating waveform may consist of the sum of several sinusoids.

- If the carrier frequency is f_c and the modulating frequencies are f_{m1} and f_{m2}, then the resulting spectrum will contain components at the frequency given by $f_c \pm i f_{m1} \pm k f_{m2}$, where i and k are integers greater than or equal to 0.

- For example, when $f_c = 100$ Hz, $f_{m1} = 100$ Hz, and $f_{m2} = 300$ Hz, the spectral component present in the sound at 400 Hz is the combination of sidebands given by the pairs: $i = 3, k = 0; i = 0, k = 1; i = 3^-, k = 2;$ and so on (see Figure 18).
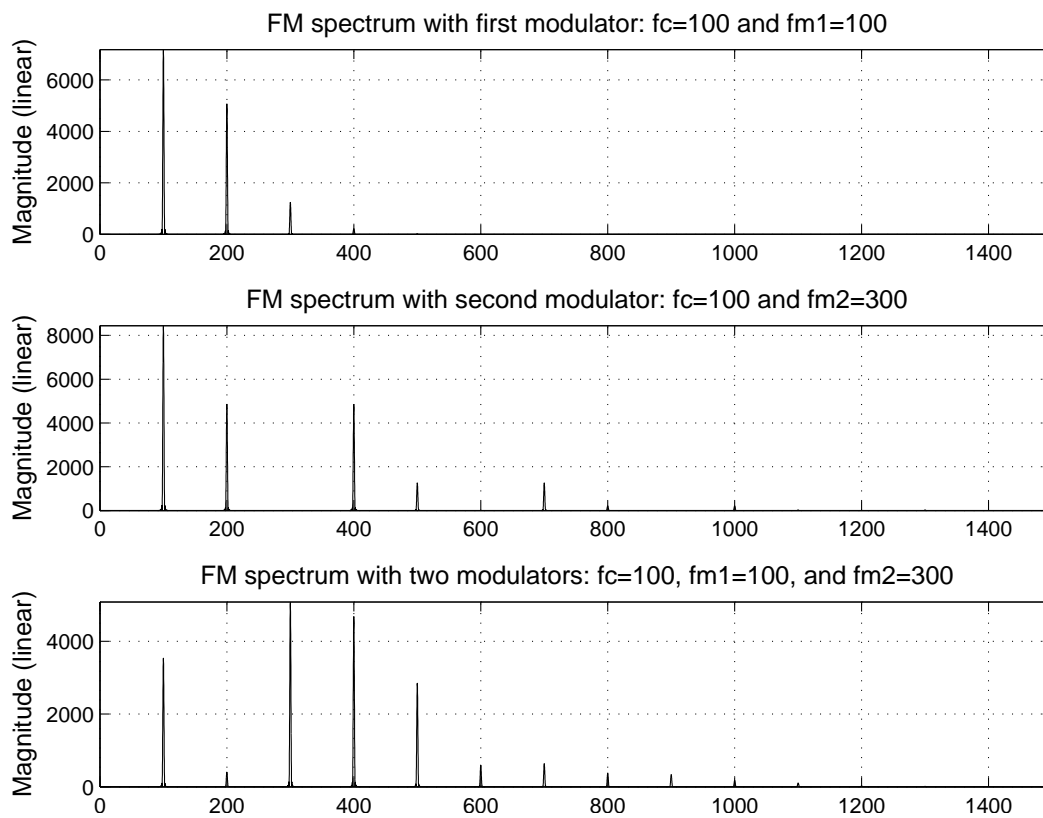
Figure 18: The FM spectrum produced by a modultor with two frequency components.

Two Modulators cont.

- Modulation indeces are defined for each component: I_1 is the index that charaterizes the spectrum produced by the first modulating oscillator, and I_2 is that of the second.

- The amplitude of the ith, k^{th} sideband $(A_{i,k})$ is given by the product of the Bessel functions

$$A_{i,k} = J_i(I_1)J_k(I_2).$$

- Like in the previous case of a single modulator, when i, k is odd, the Bessel functions assume the opposite sign. For example, if $i = 2$ and $k = 3^-$ (where the negative superscript means that k is subtracted), the amplitude is $A_{2,3} = -J_2(I_1)J_3(I_2)$.

- In a harmonic spectrum, the net amplitude of a component at any frequency is the combination of many sidebands, where negative frequencies "foldover" the 0 Hz bin (*Computer Music*).