Abstract Synthesis and Nonlinear Oscillators

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Exploring coupled behaviour

First mode:

Second mode:

Coupled behaviour:

Frequency is modulated by the oscillations of the system:

frequency deviation determined by coefficients

$$\omega = \omega_0 + d_1 \Re\{z_1\} + d_2 \Im\{z_1\} + d_3 \Re\{z_2\} + d_4 \Im\{z_1\}$$

Animations courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State

Motivation

Use of abstract synthesis:

- Better understanding and improved control of the produced sound
- Inform the inverse-problem of parameter estimation of physical systems

Outline

- The driven mass-spring oscillator
- Time variation and feedback: the nonlinear oscillator
- Power-preserving Implementation
- Loopback FM
- Feedback AM (FBAM)

The Forced Mass-Spring Oscillator

Equation of motion (forced oscillation):

$$\ddot{z} + 2\alpha\dot{z} + \omega_0^2 z = \tilde{F} \qquad \omega_0 = \sqrt{\frac{k}{m}} \quad \alpha = \frac{R}{2m} = \frac{1}{\tau}$$

Assumed unit-samplitude solution: $z = e^{j\theta}$

1st and 2nd derivative: $\dot{z} = j\dot{\theta}z \qquad \ddot{z} = (j\ddot{\theta} - \dot{\theta}^2)z$

If solution has (time-varying) amplitude:

1st and 2nd derivative:

$$\frac{d}{dt}Az = \dot{A}z + A\dot{z} = (\dot{A} + j\dot{\theta}A)z \qquad \frac{d^2}{dt^2}Az = \ddot{A}z + 2\dot{A}\dot{z} + A\ddot{z} = (\ddot{A} - A\dot{\theta}^2 + j(2\dot{\theta}\dot{A} + \ddot{\theta}A))z$$

If amplitude is a scalar: $j\dot{\theta}A \qquad (j\ddot{\theta} - \dot{\theta}^2)A$

Complex-amplitude solution: $z = A^{j\phi}e^{j\theta}$

The Mass-Spring Oscillator (Sinusoidal Driving Force):

Equation of motion: $\ddot{z} + 2\alpha\dot{z} + \omega_0^2 z = \tilde{F}$ $\qquad \tilde{F} = \frac{F}{m}e^{j\omega t}$

Assumed solution (complex samplitude): $z = A^{j\phi}e^{j\theta} \qquad \theta = \omega t \qquad \dot{\theta} = \omega$

1st and 2nd derivative:
$$\dot{z} = j\dot{\theta}z \qquad \ddot{z} = (j\ddot{\theta} - \dot{\theta}^2)z$$
$$= j\omega z \qquad = -\omega^2 z$$

The (steady-state) complex solution:
$$z = \frac{Fe^{j\omega t}}{m\underbrace{(\omega_0^2 - \omega^2 + j\omega 2\alpha)}_{Ye^{-j\phi}}} \qquad A = \frac{F}{mY}$$

$$Y = \sqrt{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega)^2}$$

The displacement: $x = \Re\{z\} = A\cos(\omega t + \phi)$

$$\phi = -\tan^{-1}\left(\frac{2\alpha\omega}{\omega_0^2 - \omega^2}\right)$$

Implementation (Bilinear Transform):

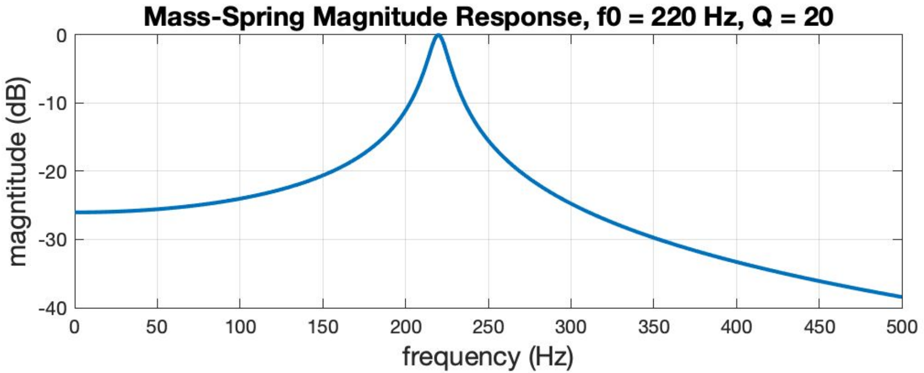
If driving force (or displacement) unknown: numerical approach

Laplace transform of the equation of motion: $s^2X(s) + 2\alpha sX(s) + \omega_0^2X(s) = F_m(s)$

Transfer function (s - domain): $H(s) = \frac{X(s)}{F_m(s)} = \frac{1}{s^2 + 2\alpha s + \omega_0^2}$

$$f_0 = \frac{\omega_0}{2\pi} \quad Q = \frac{\omega_0}{2\alpha}$$

discretization: $s \to c\frac{1 - z^{-1}}{1 + z^{-1}} \qquad c = 2f_s$



$$H(z) = \frac{X(z)}{F_m(z)} = \frac{1 + 2z^{-1} + z^2}{\underbrace{a_0 + a_1z^{-1} + a_2z^{-2}}}$$

$$a_0 = c^2 + 2\alpha c + \omega_0^2$$

$$a_1 = -2(c^2 - \omega_0^2)$$

$$a_2 = c^2 - 2\alpha c + \omega_0^2$$

System Time Variation: The Nonlinear oscillator

Feedback: The oscillation of the mass alters the system's natural frequency

Equation of motion: $\ddot{z} + 2\alpha\dot{z} + (\omega_0 + d\Re\{z\})^2 z = \tilde{F}$

Consider transfer function: $H(z) = \dfrac{1 + 2z^{-1} + z^2}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$

$$a_0 = c^2 + 2\alpha c + \omega_0^2$$
$$a_1 = -2(c^2 - \omega_0^2)$$
$$a_2 = c^2 - 2\alpha c + \omega_0^2$$

$\omega_0^2 \rightarrow (\omega_0 + d\Re\{z\})^2$

Problem: stability not guaranteed when LTI filter coefficients are made time-varying

Consider LTI filter with non-linear terms as driving force:

Equation of motion: $\ddot{z} + 2\alpha\dot{z} + \omega_0^2 z = \tilde{F} - \underbrace{(2\omega_0 d\Re\{z\} + d^2\Re\{z\}^2)z}$

Problem: driving force a function of the unknown solution. (Return later!)

The Mass-Spring Oscillator (Sinusoidal Driving Force):

Feedback: The oscillation of the mass alters the system's natural frequency

Equation of motion: $\ddot{z} + 2\alpha\dot{z} + (\omega_0 + d\Re\{z\})^2 z = \tilde{F}$

Consider previous solution: sinusoidally-driven mass-spring oscillator

displacement: $x = \Re\{z\} = \Re\left\{\frac{F}{mY}e^{j(\omega+\phi)}\right\} = A\cos(\omega t + \phi)$

$$A = \frac{F}{mY} \qquad Y = \sqrt{(\omega_0^2 - \omega^2)^2 + (2\alpha\omega)^2} \qquad \phi = -\tan^{-1}\left(\frac{2\alpha\omega}{\omega_0^2 - \omega^2}\right)$$

time-varying: ω and ω_0 → time-varying: A and ϕ

Abstract synthesis methods: Amplitude modulation (A), Phase (Frequency) modulation (ϕ)

Time-varying Oscillator via Power-Preserving Rotation

A point in the complex plane:

$$z(0) = Ae^{j\phi} = A(\cos\phi + j\sin\phi)$$

can be made to oscillate with frequency f using a power-preserving rotational matrix

$$\mathbf{r} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{power-preserving rotational matrix}} \begin{bmatrix} \Re\{z(n)\} \\ \Im\{z(n)\} \end{bmatrix} = \begin{bmatrix} A\cos\theta\cos\phi - A\sin\theta\sin\phi \\ A\sin\theta\cos\phi + A\cos\theta\sin\phi \end{bmatrix} = \begin{bmatrix} A\cos(\theta+\phi) \\ A\sin(\theta+\phi) \end{bmatrix}$$

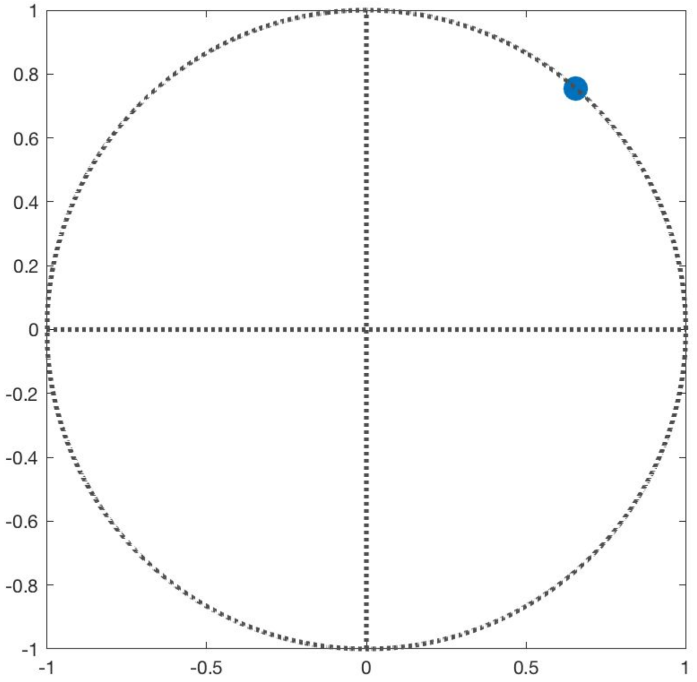
power-preserving rotational matrix $\theta = \omega T = 2\pi f / f_s$

The next time sample:

$$z(n+1) = \mathbf{r}_1 + j\mathbf{r}_2 = A\left(\cos(\theta+\phi) + j\sin(\theta+\phi)\right) = Ae^{j(\theta+\phi)} = \underbrace{e^{j\theta} z(0)}_{\text{complex multiply: sample rotation}}$$

complex multiply: sample rotation

Discrete-Time Oscillator (constant frequency)



constant frequency → ω; initial phase → ϕ

$$\left(e^{j\omega T}\right)^n A e^{j\phi} = e^{j(\omega n T + \phi)} = z(n)$$

$$e^{j\omega T} e^{j\omega T} e^{j\omega T} A e^{j\phi} = e^{j\omega T} z(2) = z(3)$$

$$e^{j\omega T} e^{j\omega T} A e^{j\phi} = e^{j\omega T} z(1) = z(2)$$

$$e^{j\omega T} A e^{j\phi} = e^{j\omega T} z(0) = z(1)$$

$$A e^{j\phi} = z(0)$$

Sample-by-sample rotation:

$$z(n) = e^{j\omega T} z(n-1)$$

Discrete-Time Oscillator (Time-Varying Frequency)

Time-varying frequency: $\omega(n)$

Sample-by-sample rotation: $z_c(n) = e^{j\omega(n)T} z_c(n-1)$

$$z_c(n) = e^{j\omega(n)T} e^{j\omega(n-1)T} e^{j\omega(n-2)T} \ldots e^{j\omega(0)T} z_c(0)$$

$$z_c(n) = e^{j\sum_{k=0}^{n} \omega(k)T} A e^{j\phi}$$

$$z_c(n) = e^{j\underbrace{\theta(n)}}$$

(Frequency modulation) instantaneous phase: $\theta(n) = \left(\sum_{k=0}^{n} \omega(k)T \right) + \phi$ **numerical solution**

If the time-varying $\omega(n)$ frequency is known:

(Phase modulation) instantaneous phase: $\theta(n) = \int_0^n \omega(n)T \; dn + C$ **analytical solution**

Discrete-Time Oscillator (time-varying frequency)

Frequency modulation (FM):

instantaneous frequency:

$$\omega(n) = \underbrace{\omega_c}_{\text{carrier frequency}} + \underbrace{d}_{\text{peak frequency deviation}} \cos(\omega_m n T)$$

sample-by-sample rotation:

$$z_c(n) = e^{j\omega(n)T} z_c(n-1)$$

Phase modulation (PM):

instantaneous phase:

$$\theta(n) = \int_0^n \omega(n)T \; dn$$

$$= \omega_c n T + \underbrace{\frac{d}{\omega_m}}_{\text{Index of modulation}} \sin(\omega_m n T) + \phi_c$$

Index of modulation $I = \dfrac{d}{\omega_m}$

analytic solution:

$$z_c(n) = e^{j\theta(n)}$$

Loopback FM

(Smyth T. and Hsu J. *SMC Malaga, Spain,* 2019)

Equation of motion: $\ddot{z} + 2\alpha\dot{z} + (\omega_0 + d\Re\{z\})^2 z = \tilde{F}$

Carrier oscillator loops ("feeds") back to modulate its own frequency:

instantaneous frequency: $\omega(n) = \omega_c + d\Re\{z_c(n)\}$

peak frequency deviation: $d = I\omega_c = B\omega_c$

loopback coefficient

sample-by-sample rotation: $z_c(n) = e^{j(\omega_c + B\omega_c\Re\{z_c(n-1)\})T} z_c(n-1)$

unit sample delay

Effects of Loopback Coefficient

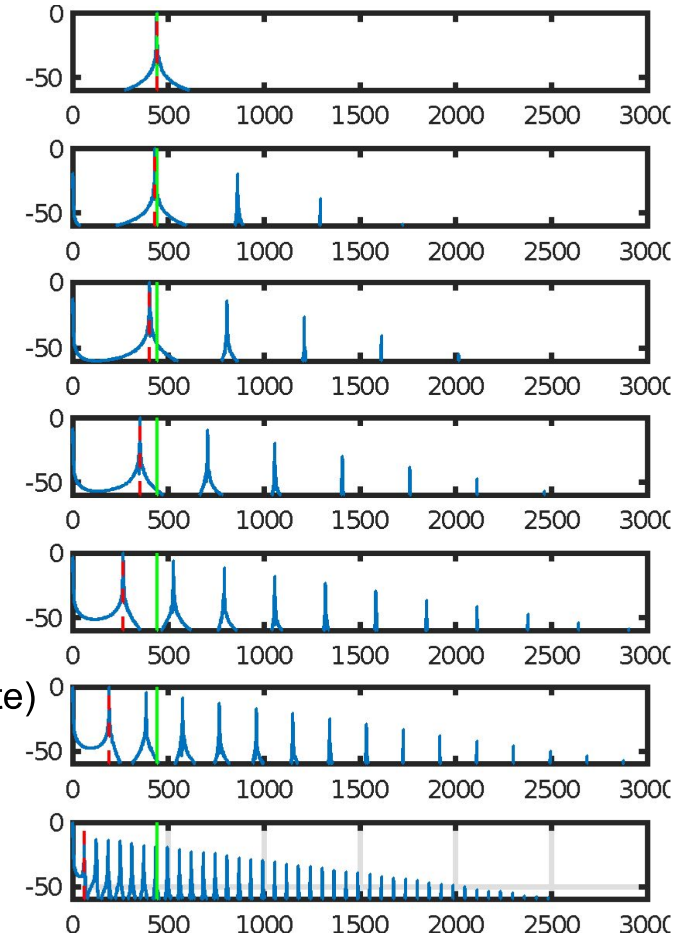
$$z_c(n) = e^{j(\omega_c + B\omega_c \Re\{z_c\underbrace{(n-1)}_{\text{unit sample delay}}\})T} z_c(n-1)$$

Increasing B (top to bottom):

- Increased spectral spread (number of harmonics)
- Lowered fundamental frequency

Implementation caveats:

- Numerical error and aliasing (dependent on sampling rate)
- Unit sample delay introduced in feedback loop
- No knowledge of sounding frequency (pitch)
- One parameter (B) influences both pitch and spectrum



Loopback FM by Phase Modulation

instantaneous frequency: $\omega(n) = \omega_c + B\omega_c \Re\{z_c(n-1)\}$

instantaneous phase:

$$\theta(n) = \int_0^n \omega(n) T \, dn + C$$

$$= \omega_c n T + B\omega_c T \Re \left\{ \underbrace{\int_0^n z_c(n-1) \, dn}_{\sum_{k=0}^{n-1} z_c(k)} \right\}$$

solve numerically? $\sum_{k=0}^{n-1} z_c(k)$

as previously shown:

$$z_c(n) = e^{j \sum_{k=0}^{n} \omega(k) T} A e^{j\phi}$$

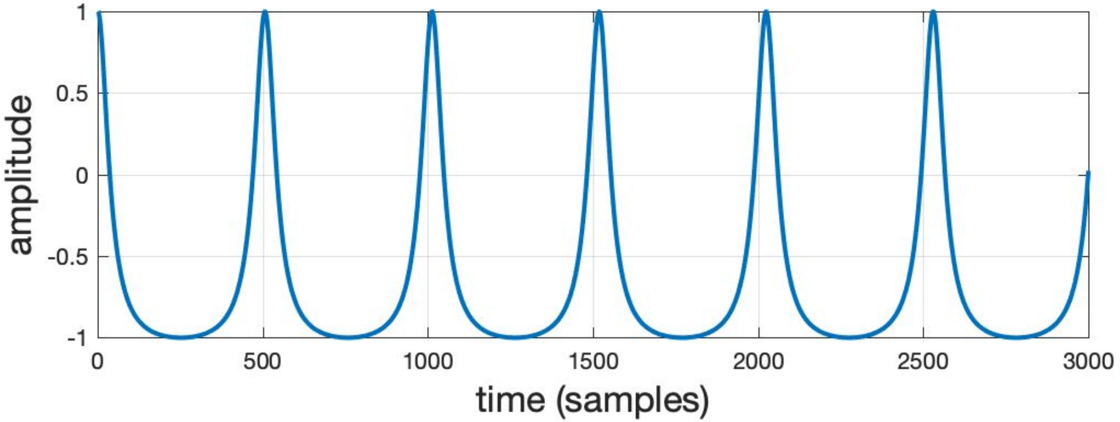
$$z_c(n) = e^{j\omega(n)T} e^{j\omega(n-1)T} e^{j\omega(n-2)T} \dots e^{j\omega(0)T} z_c(0)$$

$$z_c(n) = e^{j\omega(n)T} z_c(n-1)$$

equivalent to sample-by-sample rotation.

Loopback FM by Closed-Form Representation

The loopback FM waveform:



A periodic signal also represented by the closed form expression:

$$\Re\{z\} = \Re\left\{\frac{-b + e^{j\omega nT}}{1 - be^{j\omega nT}}\right\}$$

Solving for ω and b (Smyth T. and Hsu J. *SMC Malaga, Spain,* 2019)

instantaneous phase:

$$\theta(n) = \omega_c nT + B\omega_c T \Re\underbrace{\left\{\int_0^n z_c(n-1)\ dn\right\}}_{\Re\left\{\int_0^n \frac{-b+e^{j\omega nT}}{1-be^{j\omega nT}}\right\}}$$

solve analytically (expression omitted)

Second expression for phase:

$$\theta(n) = \angle\left(\frac{-b+e^{j\omega nT}}{1-be^{j\omega nT}}\right) = \omega nT - 2\tan^{-1}\left(\frac{-b\sin(\omega nT)}{1-b\cos(\omega nT)}\right)$$

Make expressions equal and solve:

$$\omega = \omega_c\sqrt{1-B^2} \qquad b = \frac{1-\sqrt{1-B^2}}{B} \quad 0 < b < 1$$

Loopback FM Modal Synthesis Sounds

KICK	
SNARE	
TOM TOM	
CIRCULAR PLATE	
LOOPBACK FM BEAT	

Loopback FM: instantaneous frequency

Assumed solution:

$$z = \frac{-b + e^{j\omega t}}{1 - be^{j\omega t}} = e^{j\theta}$$

phase:

$$\theta = \omega t - 2\tan^{-1} x_b \qquad x_b = \frac{-b\sin(\omega t)}{1 - b\cos(\omega t)}$$

Instantaneous frequency (derivative of phase):

$$\dot{\theta} = \omega - 2\frac{\dot{x}_b}{1 + x_b^2} \qquad \dot{x}_b = \frac{-\omega b(\cos(\omega t) - b)}{(1 - b\cos(\omega t))^2}$$

$$= \omega + \frac{2b\omega(\cos(\omega t) - b)}{1 - 2b\cos(\omega t) + b^2} \qquad B = \frac{2b}{1 + b^2}$$

$$= \omega + \frac{B\omega(\cos(\omega t) - b)}{1 - B\cos(\omega t)} \qquad b = \frac{1 - \sqrt{1 - B^2}}{B} \qquad 0 < b < 1$$

$$= \frac{\omega_c(1 - B^2)}{1 - B\cos(\omega t)} \qquad \omega_c = \frac{\omega}{\sqrt{1 - B^2}}$$

Loopback FM: frequency as a function of itself

Instantaneous frequency:

$$\dot{\theta} = \frac{\omega_c(1 - B^2)}{1 - B\cos(\omega t)}$$ adding and subtracting $B\omega_c \cos(\omega t)$ to/from the numerator

$$= \frac{\omega_c(1 - B\cos(\omega t)) + B\omega_c(\cos(\omega t) - B)}{1 - B\cos(\omega t)}$$

$$= \omega_c + \omega_c B \frac{\Re\{-B + e^{j\omega t}\}}{\Re\{1 - Be^{j\omega t}\}}$$ $$B = \frac{2b}{1 + b^2}$$ **FBAM (discussed next)**

$$= \omega_c + \omega_c B \Re\left\{\frac{-b + e^{j\omega t}}{1 - be^{j\omega t}}\right\}$$

$$= \omega_c + \omega_c B \Re\{z\}$$

Feedback AM (FBAM)

Original FBAM in discrete time: (Kleimola J. et al. *EURASIP,* 2011)

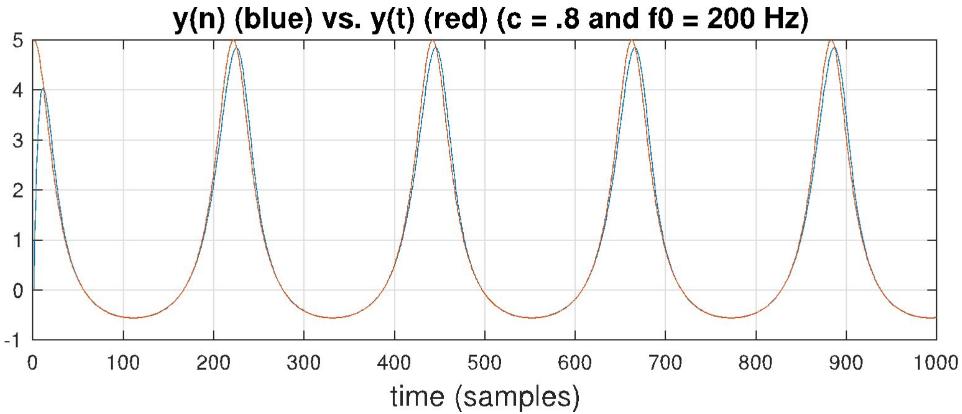
$$y(n) = \underbrace{\cos(\omega_0 nT)}_{\text{input}}(1 + cy\underbrace{(n-1)}_{\text{unit sample delay}})$$

FBAM in continuous time (closed form representation): (Smyth T. *AES New York,* 2019)

input: $x(t) = \cos(\omega_0 t) = \Re\left\{e^{j\omega_0 t}\right\}$

output:
$$\begin{aligned} y &= x(1 + cy) \\ &= \frac{x}{1 - cx} \\ &= \frac{\Re\{e^{j\omega t}\}}{1 - c\Re\{e^{j\omega t}\}} \end{aligned}$$

Normalized FBAM Equals Loopback FM



peak: $A_p = \frac{1}{1-c}$

valley: $A_v = -\frac{1}{1+c}$

amplitude: $A_{vp} = A_p - A_v = \frac{2}{1-c^2}$

Normalizing (and offsetting) output: $y_n = 2\left(\frac{y - A_v}{A_{vp}}\right) - 1 \quad = \frac{\Re\{-c + e^{j\omega t}\}}{\Re\{1 - ce^{j\omega t}\}}$

Loopback FM $= \Re\left\{\frac{-b + e^{j\omega t}}{1 - be^{j\omega t}}\right\}$

Loopback FM, FBAM and (Nonlinear) Equation of Motion

Equation of motion: $\ddot{z} + 2\alpha\dot{z} + (\omega_0 + d\Re\{z\})^2 z = \tilde{F}$

Assume solution:

$$\underbrace{\ddot{z} + 2\alpha\dot{z} + \omega_0^2 z}_{\underbrace{(j\ddot{\theta} - \dot{\theta}^2 + j2\alpha\dot{\theta} + \omega_0^2)}_{\tilde{Y}} z} = \tilde{F} - \underbrace{(2\omega_0 d\Re\{z\} + d^2\Re\{z\}^2)z}_{g}$$

$$z = e^{j\theta}$$

$$\dot{z} = j\dot{\theta}z$$

$$\ddot{z} = (j\ddot{\theta} - \dot{\theta}^2)z$$

$$\dot{\theta}_b = \omega_0 + \omega_0 B\Re\{z\}$$

$$\ddot{\theta} = \omega_0 B\Re\{\dot{z}\}$$

$$= -\omega_0 B\dot{\theta}\Im\{z\}$$

$$\tilde{Y}_r z = (-\dot{\theta}^2 + \omega_0^2)z = g \quad B = d$$

$$\tilde{Y}_i z = j(\ddot{\theta} + 2\alpha\dot{\theta})z = \tilde{F}$$

$$H(z) = \frac{X(z)}{F_m(z)} = \frac{1 + 2z^{-1} + z^2}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

Summary

The driven mass-spring oscillator with feedback:

- A nonlinear oscillator producing time-varying amplitude and frequency and spectrum
- Implemented by power-preserving loopback FM and “equivalent” feedback AM

Future Work

Abstract synthesis representations:

- Better understanding and improved control of the produced sound
- Inform the inverse-problem of parameter estimation of physical systems
- CURRENT: the human voice: synthesis and estimation of the glottal pulse