

Spectral Distortion Using Second-Order Allpass Filters

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ABSTRACT

This work presents a technique for detuning or applying phase distortion to specific spectral components of an arbitrary signal using a cascade of parametric second-order allpass filters. The parametric second-order allpass provides control over the position and slope of the transition region of the phase response, and this control can be used to tune a phase distortion effect to a specific frequency range. We begin by presenting the phase response of a cascade of first-order filters, which we relate to that of the parametric second-order allpass. Time-varying parameters and the time-varying phase response are derived for the second-order case, and we provide examples demonstrating the frequency-selective phase distortion effect in the context of processing of instrumental sounds.

1. INTRODUCTION

Allpass filters are a fundamental synthesis building-block, and have many applications in computer music. Allpass filters have been studied with applications to both synthesis and effects processing, often in cascaded form or with modulation of the filter coefficients. In [1], the dispersive effects of a cascade of first-order allpass filters are exploited to produce a frequency-dependent delay effect, called a spectral delay filter. Kleimola et al., in [2], propose the use of a cascade of filters, with audio-rate modulation of coefficients, to obtain complex AM- and FM-like spectra, with applications to synthesis, physical modeling, and effect processing. The dispersive effects of cascaded allpass filters have been used in physical modeling of piano strings [3] and spring reverberators [4]. Finally, Lazarini et al. describe the use of a first-order allpass filter in phase distortion synthesis [5], where the authors modulate the filter coefficient with a modulation function designed to create a desired time-varying phase shift.

In this paper, we describe the use of cascaded parametric second-order allpass filters in detuning and phase distortion applications. The parametric second-order allpass used here has some important advantages over the first-order, as it offers greater control over the phase transition region - the range of frequencies where the largest amount

of phase distortion will occur. Instead of directly modulating coefficients, we focus on modulating the filter parameters which control the placement and size of this transition region. This makes it possible to apply phase distortion to specific spectral bands, independently of others.

This work is part of a larger investigation into the use of cascaded allpass filters in generative self-oscillating feedback systems. By making the phase response of the feedback system time-varying, it is possible to avoid the static timbres characteristic of some feedback systems, and introduce more dynamic musical behaviors. Though the applications discussed in this work do not involve feedback systems, the techniques introduced here are a first step toward that goal.

In Section 2 of this paper, we derive the first-order allpass filter, and the phase response of a cascade of first-order filters. We also relate the first-order filter to the second-order allpass filter used in this work. In Section 3, we describe how the filter parameters and phase response can be made time-varying. A basic example of the detuning/phase distortion effect is provided. Applications to synthesis and processing of instrumental sounds are presented in Section 4.

2. THE ALLPASS FILTER

It is well known that an allpass filter has unity gain at all frequencies. It is, therefore, frequently used in situations where a frequency-dependent phase shift is desirable, without imparting any gain or attenuation.

2.1 First-Order Allpass Filter

A first-order allpass filter, given by

$$H_1(z) = \frac{c + z^{-1}}{1 + cz^{-1}}, \quad (1)$$

has a single pole and zero at $-c$ and $-1/c$, respectively, where $|c| < 1$ for stability. Because of the stability constraint, the pole at $-c$ lies inside the unit circle, while the reciprocal zero lies outside the unit circle (making the filter maximum phase). If the coefficient c is real, both pole and zero will lie on the real axis, with $|c|$ controlling the spacing of the pole-zero pair from the unit circle—a magnitude of c close to 1 positions the pole and its reciprocal zero closer to (as well as more equidistant from) the unit circle (see Fig. 1 for example of $c = 0.9$ and $c = -0.6$).

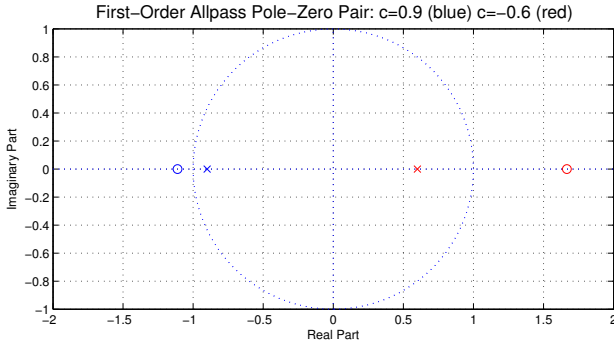


Figure 1. The position of the pole-zero pair for the first-order allpass filter (1). A coefficient of $c = 0.9$ yields the pair on the real axis to the left (blue) and $c = -0.6$ yields the pair to the right (red).

The magnitude of (1), is given by:

$$|H_1(\omega)| = \left| \frac{c + e^{-j\omega T}}{1 + ce^{-j\omega T}} \right| = 1, \quad (2)$$

where f_s is the sampling rate and $T = 1/f_s$ is the sampling period. A sampling rate of $f_s = 44100$ Hz was used throughout this paper.

The phase of (1) is given by

$$\begin{aligned} \angle H_1(\omega) &= \frac{\angle(c + e^{-j\omega T})}{\angle(1 + ce^{-j\omega T})} \\ &= \angle e^{-j\omega} + \angle(1 + ce^{j\omega}) - \angle(1 + ce^{-j\omega}) \\ &= -\omega + 2 \tan^{-1} \left(\frac{c \sin(\omega)}{1 + c \cos(\omega)} \right), \end{aligned} \quad (3)$$

where the final result is obtained using the fact that $\angle z = \tan^{-1}(\Im(z)/\Re(z))$. Since the denominator inside the $\tan^{-1}(\cdot)$ is always positive (for $|c| < 1$), and the numerator can change sign, the contribution due to the $\tan^{-1}(\cdot)$ term has a possible range of $\pm\pi/2$ rad (and $\pm\pi$ rad for $2 \tan^{-1}(\cdot)$), and thus contributes an oscillation around the first linear-phase term [6]. The result, as shown in Figure 2, is a phase response that is monotonically decreasing, with an overall decrease of 2π rad as ω increases by 2π rad / sample.

Rearranging (3) yields the following expression for the coefficient:

$$c = -\frac{\tan\left(\frac{\angle H_1 + \omega}{2}\right)}{\tan\left(\frac{\angle H_1 + \omega}{2}\right) \cos(\omega) - \sin(\omega)}, \quad (4)$$

which, for $\angle H_1 = -\pi/2$, conveniently reduces to

$$c = \frac{\tan(\omega/2) - 1}{\tan(\omega/2) + 1}. \quad (5)$$

That is, the behaviour of the phase response can be controlled to some extent using (5), by specifying the angular frequency

$$\omega = 2\pi \frac{f_{\pi/2}}{f_s} \text{ rad/sample}, \quad (6)$$

where $f_{\pi/2}$ is the frequency in Hz at which 90° ($\pi/2$) phase shift is reached.

A general higher-order all-pass filter can be made by cascading several first-order allpass sections

$$H_k(z) = \prod_{k=0}^K \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}, \quad (7)$$

where if the allpass filter has real coefficients, for each complex root a_k , there must be a corresponding complex conjugate root a_k^* , making the phase anti-symmetric about $\omega = 0$ [6]. The phase response for the overall filter is the sum of the phases for each section, and is given by

$$\angle H_k(\omega) = -K\omega - 2 \sum_{k=1}^K \tan^{-1} \left(\frac{R_k \sin(\omega - \theta_k)}{1 - R_k \cos(\omega - \theta_k)} \right), \quad (8)$$

for $a_k = R_k e^{j\theta_k}$. In the following, we consider a special second-order case.

2.2 Second-Order Allpass Filter and its Cascade

Though there is some control over the behaviour of the first-order filter by using (5) to specify the frequency $f_{\pi/2}$ at which the phase response is -90° , more control is afforded using a special case of the second-order allpass filter, for which there is an additional ‘‘bandwidth’’ parameter [7].

The transfer function of a second-order allpass filter may be expressed using (7) for $k = 2$, but a more convenient formulation is given in [7] by:

$$H_2(z) = \frac{-c + d(1-c)z^{-1} + z^{-2}}{1 + d(1-c)z^{-1} - cz^{-2}}, \quad (9)$$

which allows for specification of coefficients

$$d = -\cos\left(\frac{2\pi f_{\pi}}{f_s}\right) \quad (10)$$

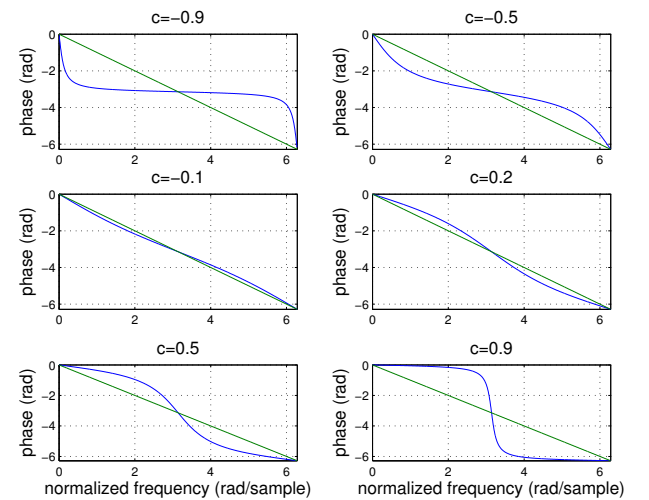


Figure 2. The phase response (blue), monotonically decreasing by 2π with an increase in ω of 2π , is shown with the linear-phase term $-\omega$ from (3).

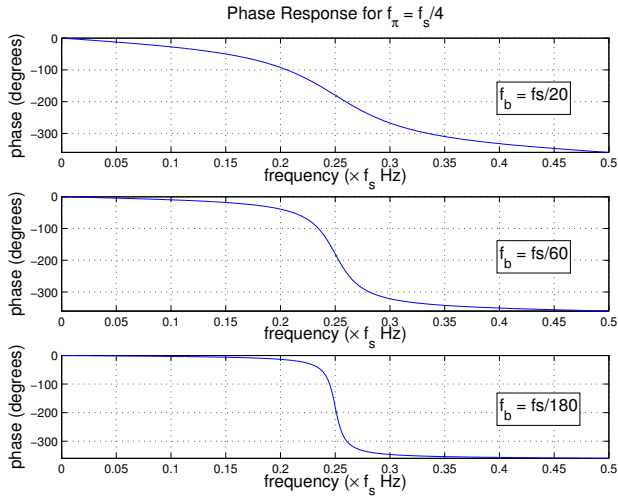


Figure 3. The phase response of the second-order allpass filter with $f_\pi = f_s/4$ and with varying bandwidths f_b .

and

$$c = \frac{\tan(\pi f_b/f_s) - 1}{\tan(\pi f_b/f_s) + 1} \quad (11)$$

according to the frequency f_π (in Hz) at which the phase response is -180° (or $-\pi$), and a bandwidth of the phase transition region f_b .

Figure 3 shows an example of the monotonically decreasing phase response where $f_\pi = f_s/4$ and several values are used for f_b . The curves begin with a gentle change in phase, followed by an increased downward slope reaching -180° at f_π , before tapering off again to a more gentle slope toward maximal delay. Adjusting f_π and f_b allows for both placement of the frequency point at which a 180° phase shift is reached, and control over the slope of the phase transition region.

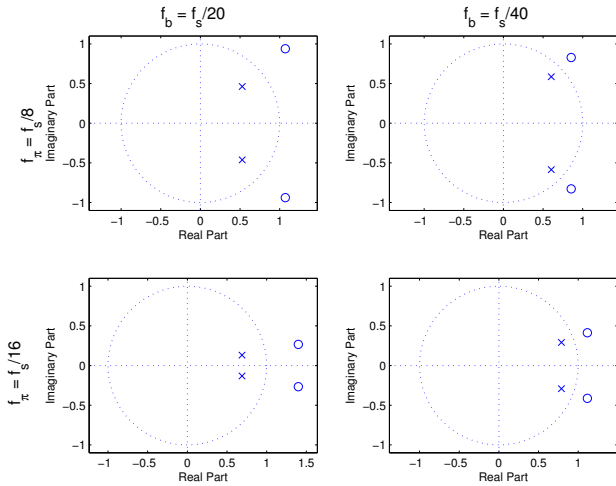


Figure 4. The position of the poles and zeros for the second-order allpass filter described by (9). Holding the parameter f_π constant for each row, and f_b constant for each column, we can see how the former adjusts the angle of the two pole-zero pairs, while the latter controls their distance to the unit circle.

The effect of the f_π parameter can also be seen in Figure 4 by how it adjusts the angle of the two pole-zero pairs on the unit circle. The bandwidth f_b controls the distance of the pole and zero to the unit circle.

3. PHASE DISTORTION WITH THE SECOND-ORDER ALLPASS

Through careful tuning of second-order allpass filter (9), it is possible to apply a time-varying phase distortion effect to a specific band of the spectrum.

3.1 A time-varying allpass filter

In order to make (9) time varying, it is necessary to redefine coefficients d and c as functions of time. In this work, rather than modulating the coefficients directly, it is the parameter f_π that is made time varying:

$$\tilde{f}_\pi(n) = f_\pi + M \cos\left(\frac{2\pi f_m n}{f_s}\right), \quad (12)$$

where $\tilde{\cdot}$ indicates a function made time varying, f_π is as previously defined, M is the depth of modulation, f_m is the modulation frequency, and n is the discrete time index. Here, \tilde{f}_π is modulated sinusoidally (though this is not a requirement), and can be seen as an FM signal, with f_π being the carrier frequency (which it will be subsequently called when referred to in the time-varying case).

The coefficient d from (10) is then replaced with

$$\tilde{d}(n) = -\cos\left(\frac{2\pi \tilde{f}_\pi(n)}{f_s}\right), \quad (13)$$

yielding the filter's difference equation

$$y(n) = -cx(n) + \tilde{d}(n)(1-c)x(n-1) + x(n-2) - \tilde{d}(n)(1-c)y(n-1) + cy(n-2). \quad (14)$$

Expressing (9) as the difference equation in (14) follows the example of [2], in which filter output $y(n)$ is a combination of delayed versions of input $x(n)$ which are ring modulated with sinusoidally-varying coefficients. Here, in contrast, coefficient *parameters* are sinusoidally modulated yielding time-varying coefficients that are effectively FM signals as shown in (13). Following (8) for the phase of the general allpass, it can be easily shown that the time-varying second-order allpass has a family of phase responses given by

$$\theta_A(\omega, n) = -2\omega + 2 \tan^{-1} \left[\frac{\tilde{d}(n)(1-c) \sin(\omega) - c \sin(2\omega)}{1 + \tilde{d}(n)(1-c) \cos(\omega) - c \cos(2\omega)} \right] \quad (15)$$

To gain some intuition for the effect of this new time-varying parameter $\tilde{f}_\pi(n)$, consider the effect of the parameters f_π , f_b , and M on this group of phase responses. Figure 5 illustrates an example for $f_\pi = f_s/4$ Hz, $f_b = f_s/40$

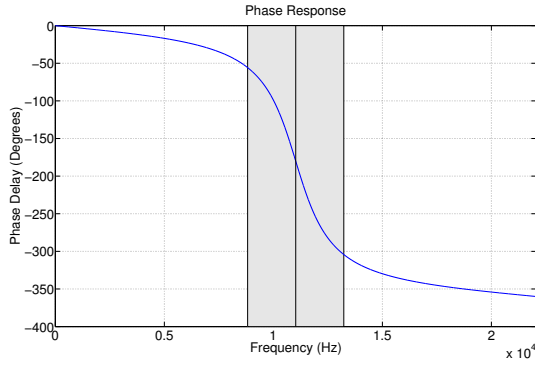


Figure 5. Effect of modulation depth and transition region on phase distortion. Vertical shaded region indicates range of $\tilde{f}_\pi(n)$, i.e. $f_\pi - M \leq \tilde{f}_\pi(n) \leq f_\pi + M$.

Hz, and $M = f_s/10$. Recalling (12), we can imagine the transition region centered on $\tilde{f}_\pi(n)$ being shifted up and down in frequency - left and right in Figure 5 - at a rate corresponding to f_m . The upper and lower limits of this shift are provided by M , and indicated by the shaded box in Figure 5. Spectral components in the shaded region will experience significant time-varying phase shift as f_π is modulated, while components outside of that region will experience relatively less. Components below the transition region will be delayed by a small and relatively stable amount, while those above the transition region will be delayed by a larger, but still relatively stable amount. By placing f_π at some frequency of interest, and tuning f_b and M to generate the appropriate transition region, we can apply phase distortion to components which fall into the transition region, while leaving others (relatively) unmodified.

Given a desired frequency deviation $|\tilde{f}_\pi(n) - f_\pi$, there is a dependency between M and f_m . This can be explained by the interaction between the modulation frequency f_m and modulation index M in determining the instantaneous frequency of an FM signal. Following [8], and assuming a constant modulation index M , the instantaneous frequency of \tilde{f}_π is given by:

$$\tilde{f}_\pi(n) = f_\pi - M f_m \sin(2\pi f_m n + \phi_m) \quad (16)$$

By substituting a cosine modulation function (which allows us to disregard time, since the cosine will begin at maximum deviation), and rearranging (16) to solve for M , we obtain:

$$M = \frac{|\tilde{f}_\pi| - f_\pi}{f_m} \quad (17)$$

where $|\tilde{f}_\pi|$ is the desired peak frequency deviation, and f_π and f_m are the carrier frequency and modulation frequency as defined above. Equation (12) then becomes:

$$\tilde{f}_\pi(n) = f_\pi - M f_m \cos\left(\frac{2\pi f_m n}{f_s}\right) \quad (18)$$

As shown in (8), a cascade of identical allpass filters will produce an overall phase response that is the sum of the phases of each section. In addition, the composite filter

will have a phase response with a similar curve to one section, with the only difference being a greater range between minimum and maximum delay (the range increasing by a factor of K , the cascade length). This is an important consideration, as the cascade length corresponds to the maximum possible amount of phase distortion. It is often necessary to adjust the cascade length to obtain the desired amount of distortion. In the following discussion, K is the cascade length in terms of second-order filters.

The bandwidth of a modulated cascade is similar to that of FM synthesis, but with a different dependency on the modulation index M . Whereas for classical FM synthesis, the bandwidth can be approximated by

$$BW_{f_m} = 2(M + 1)f_m, \quad (19)$$

the approximate maximum bandwidth of a modulated second-order allpass is given by:

$$BW_{ap} = 2(M + 2)f_m \quad (20)$$

Thus, given an input signal with a single component with frequency f_0 , the resulting spectrum will consist of

$$f_0 \pm k f_m, \text{ where } k = \{0, 1, \dots, M + 2\}. \quad (21)$$

The cascade length K has a small effect on the overall bandwidth, by introducing additional, smaller amplitude sidebands. It has been shown that cascades of allpass filters can become unstable due to numerical error when larger values of K are used [9], so K generally should not be used as a bandwidth parameter. Both f_b and K affect the relative amplitudes of f_0 and the generated sidebands. In the case of a single second-order allpass filter, as f_b decreases, the amplitudes of the sidebands increase while that of f_0 decreases. As K increases, the amplitudes are affected in a more complex manner, due to the recursive processing at each stage in the allpass cascade. If each allpass stage is modulated at the same rate, the output of each stage will contain sidebands at the same frequencies. The generated sidebands sum to produce more complex spectra, with slightly larger bandwidths, as further sidebands are generated around components from earlier allpass stages. It is important to note that the above measures only apply when input components fall into the transition region controlled by f_b . Input components which fall to either side of this region will be affected to a lesser extent or hardly at all, as described above.

In summary, the parameters of the modulated second-order filter cascade are as follows:

- f_π : modulated filter “carrier” frequency - controls placement of frequency band affected by modulation.
- f_b : filter “bandwidth” - affects width of frequency band affected by modulation, and amplitudes of sidebands.
- f_m : modulation frequency - controls spacing of sidebands at audio rate / speed of vibrato at sub-audio rates.

- M : modulation depth - controls maximum bandwidth of spectrum at audio rate / depth of vibrato at sub-audio rates.
- K : filter cascade length - affects amount of phase distortion applied to frequencies in phase transition region, also affects overall bandwidth. Large K can lead to instability.

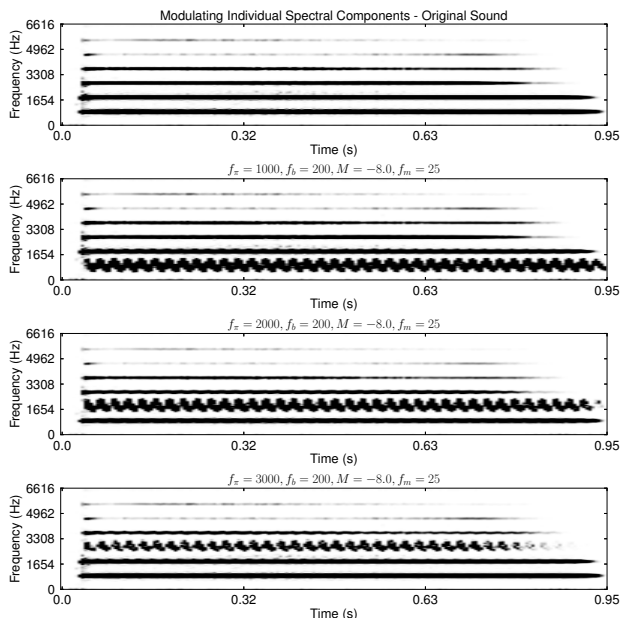


Figure 6. Modulating individual components of clarinet tone. From top to bottom: spectrograms of the original signal and processed versions in which frequency modulation was applied to the fundamental, first, and second harmonics, respectively. In all cases, $\tilde{f}_\pi(n) = f_\pi - M f_m \cos(2\pi f_m n / f_s)$, $K = 10$.

3.2 Simple Application to a Clarinet Sample

To demonstrate the frequency-selective nature of this effect, consider a clarinet tone, which contains primarily odd harmonics. Figure 6 shows the effect of a modulated all-pass cascade on the spectrum. The all-pass cascade has been tuned to affect only specific harmonics of the clarinet sound. The carrier frequency f_π is set to the frequency of the i th harmonic and the bandwidth f_b is set to 200 Hz. This creates a narrow transition region with a steep slope centered around the frequency of the harmonic. The carrier f_π is modulated by a 25 Hz sinusoid.

4. APPLICATIONS

As a basic signal processing effect, it is possible to use this technique to animate the spectra of steady-state tones (as in the clarinet example of above). As shown previously in Figure 6, specific spectral components can be modulated independently. This effect could be useful to add interest to otherwise static timbres, perhaps as a post-processing stage applied to common “analog” waveforms. Here we discuss other musical applications.

4.1 Modulation at Sub-Audio Rates

As described above, through careful tuning of the filter parameters, it is possible to apply a frequency modulation effect to specific frequency ranges independently of others. With sub-audio coefficient modulation rates, this produces a selective vibrato effect. Various partials can be modulated independently of the rest, as illustrated in Figure 7. A recording of a clarinet improvisation was processed with a cascade of second-order allpass filters, adding a $f_m = 2$ Hz vibrato to a selected portion of the spectrum. The filter carrier frequency f_π was set to 3674 Hz (the visual midpoint of the spectrum), and a wide filter bandwidth $f_b = 800$ was used in order to affect a range of frequencies. The modulation depth M was set to 300, producing a 600 Hz swing for $\tilde{f}_\pi(n)$ (17), and it was necessary to use a filter cascade of length $K = 15$ in order to obtain the amount of phase distortion necessary to produce dramatic changes in frequency. Referring back to Figures 3 and 5, we see how f_b affects the slope of the phase transition region. As f_b is increased, producing a more shallow slope, the amount of phase distortion applied to any particular frequency component will decrease. By increasing K , we can compensate for this by increasing the maximum phase delay of the system - and consequently the phase delay applied to any given input component.

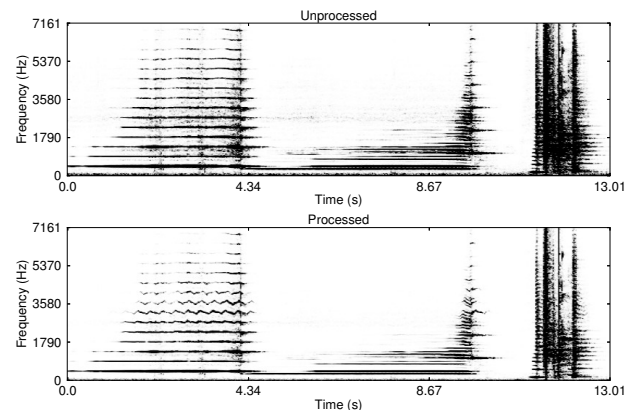


Figure 7. Clarinet passage with spectral modulation of selected harmonics for $f_\pi = 3674$ Hz, $M = 300$, $f_b = 800$ Hz, $f_m = 2$ Hz. Length of allpass cascade $K = 15$.

4.2 Audio-Rate Phase Distortion

In addition to sub-audio modulation, it is possible to modulate the filter parameters at audio rates. As described in Section 3, this has the effect of building FM-like sidebands around component frequencies present in the original signal because of the ring-modulation and FM terms in (14), and therefore can produce very rich spectra. Figure 8 provides a spectrogram of a clarinet performance through a single cascade of identical second-order allpass filters, which are driven by a fundamental frequency estimator. The carrier frequency f_π is set to the estimated fundamental.

In this case, the upper harmonics of the sound are left relatively unmodified, while the lower components (those

nearest to the estimated fundamental frequency) are significantly modulated. FM-like sidebands appear around at 100 Hz intervals around the distorted frequencies. The amplitude envelopes of new components follow those of the originals. Temporal aspects are also preserved, but a smearing effect is added. There is a possible trade-off that may need to be considered as greater cascade lengths produce a more pronounced “smearing” effect.

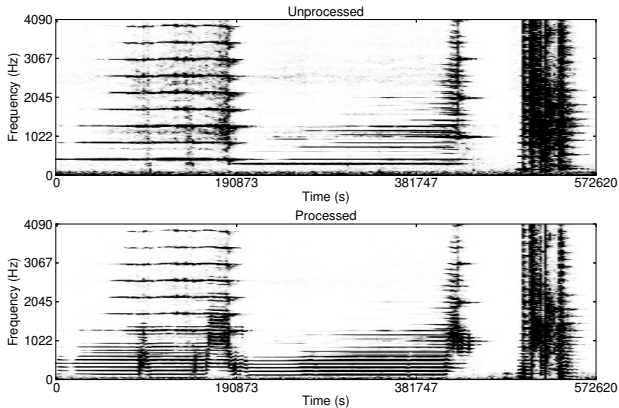


Figure 8. Clarinet passage with $\tilde{f}_\pi(n)$ driven by fundamental frequency estimator. Here, $f_m = 100$ Hz, $M = 1$, $f_b = 500$ Hz, and f_π is the estimated fundamental. Cascade length is $K = 5$.

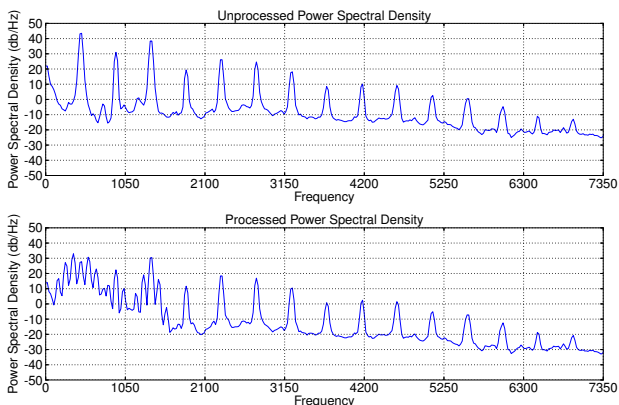


Figure 9. Example of audio-rate modulation of $\tilde{f}_\pi(n)$ on clarinet tone. Parameters used are the same as those in Figure 8.

Figure 9 provides another view of this effect on a portion of the clarinet passage used above. The excerpt used is approximately the first sonority of Figure 8 (approximately the first 4 seconds). Here, we see spectra of both the distorted and undistorted clarinet tone. The estimated fundamental frequency and a few of the surrounding components exhibit significant sidebands, while the remainder of the spectrum is left relatively unmodified. The effect of the interaction between M and f_b is to control the range of affected components (see Figure 5).

The spectrum of the modulated tone is dense, and contains subharmonics not present in the original signal. The overall spectral envelope follows that of the original tone.

5. CONCLUSION

Cascaded and coefficient-modulated second-order allpass filters have useful applications in detuning and phase distortion applications. The second-order allpass provides a greater amount of control over the transition region of the phase response than does the first. This property can be used to apply phase distortion to only specific frequency ranges. The parametric second-order allpass filter was presented, along with a means of making the parameters - and therefore the coefficients - time-varying. This allows for the use of the filter as a frequency-selective phase distortion effect, and a simple example of this effect applied to a clarinet tone was provided. Some more realistic example applications were also provided. The first example demonstrated a low-frequency detuning effect applied to the upper harmonics of a recorded clarinet passage, and the second applied a high-frequency phase distortion to the same passage.

These filters are also being studied as components in self-oscillating feedback systems, where they provide a method of avoiding static timbres without introducing unwanted gain or attenuation into the system. By introducing a time-varying, frequency-dependent phase shift, the system function changes over time, thus producing dynamic and evolving sonic behavior.

This technique could also be extended to modulation of the filter bandwidth parameter f_b , the use of non-sinusoidal parameter modulation functions, and the use of second-order allpass cascades as a synthesis technique - whether driven by a sinusoid or some other signal. Finally, multiple cascades with different time-varying parameters could be used in series, applying differing amounts of phase distortion to various spectral bands.

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