DISCRETE-TIME SIMULATION OF AIR-FLOW CUT-OFF IN PRESSURE-CONTROLLED VALVES

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ABSTRACT

In this research, the behaviour of the differential equation governing volume flow through a pressure-controlled valve is examined with particular attention given to the rather troublesome transition between an open and closed valve. A closed-form solution for the time evolution of volume flow is given and used to derive an update for the volume flow. The result is a smooth, nearly alias free transition between the two states. The form of the update is similar to that of a leaky valve where the leakage decreases as the volume flow decreases.

1. INTRODUCTION

Pressure controlled valves exist in mechanically driven musical instruments such as brasses and woodwinds and are also found in many biological sound producing mechanisms such as the vocal folds in the human larynx and the vibrating membrane in the songbird's vocal organ, the syrinx. Aliasing is a common problem in physical modeling synthesis, and in particular in models with a pressure controlled valve where a membrane or reed has the ability to close completely. Depending on how this event is handled, an abrupt termination of air flow in a closed valve can create undesirable discontinuities among other artifacts.

Relatively low audio sampling rates can create situations where, in the worst case, the model becomes completely unstable when discretized using an algorithm whose accuracy is dependent on the sampling period. In some instances this is remedied by moving to a higher-order-error algorithm such as the trapezoid rule for numeric integration [1, 2, 3].

Much of the high frequency content circulating in a system involving a valve and an acoustic tube will be removed through the use of reflection and wall attenuation filters, which, lowpass by nature, tend to reduce the effects of mild aliasing. In some cases however, a model may seem satisfactorily stable yet there will still be evidence of aliasing components in the output spectrum which can only be removed at the source.

2. A MODEL FOR A PRESSURE CONTROLLED VALVE

There are three possible configurations for the motion of a pressure-controlled valve in acoustic tubes [4]: 1) the valve is blown closed (as in the case of woodwind instruments), 2) the valve is blown open (as in the case of the human larynx and lip reed instruments) and 3) the valve moves perpendicular to the direction of air flow (as in the case of the avian syrinx) [5]. Though valves will often use combinations of these configurations, one will usually predominate and make a satisfactory approximation to the valve's overall motion [2].

As this research was developed while modeling the avian syrinx, the digital simulation of air flow through the valve is presented in the context of the *transverse configuration*. The differential equations describing air flow through other valve configurations are very similar however, and therefore also may make use of the results presented here.

The airway in the songbird consists of a trachea which divides into the left and right bronchus at its base. At the top of each bronchus, just below the junction with the trachea, a flexible membrane forms a constriction (or pressure-controlled valve) with varying heights in the bronchial lumen (see Figure 1) [6, 7, 2, 3].

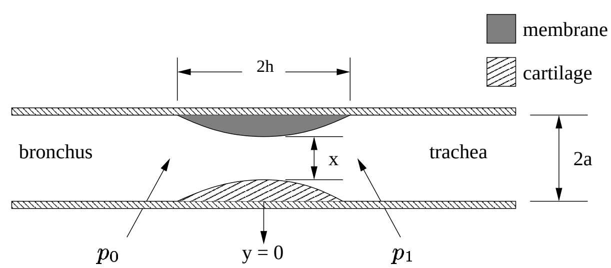

Figure 1: *The transverse model of a pressure controlled valve.*

During voiced song, the membrane is set into motion by air flow, vibrating at a frequency determined partly by its mass and tension, and partly by the resonance of the air column to which it is connected [7]. The model of the valve displacement and the resulting pressure through the constriction is based on the mechanical properties of the membrane and the Bernoulli equation for the air flow. The methods used for digitally simulating the avian vocal tract model are more thoroughly described in [3] and [2].

The model has the following four key variables which vary over time during sound production:

p_0	$\triangleq$	pressure on the bronchial side of the constriction
U	$\triangleq$	air volume flow through the syrinx
x	$\triangleq$	displacement of the membrane
p_1	$\triangleq$	pressure on the tracheal side of the constriction

Figure 2 shows the time evolution of the four variables in response to an example applied bronchus pressure. When the valve

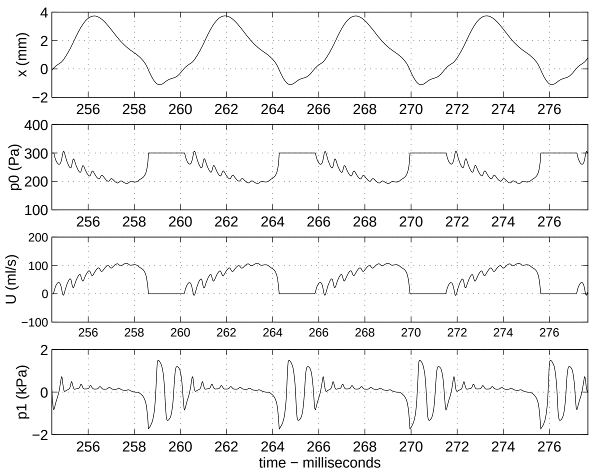

Figure 2: *Model variable waveforms.*

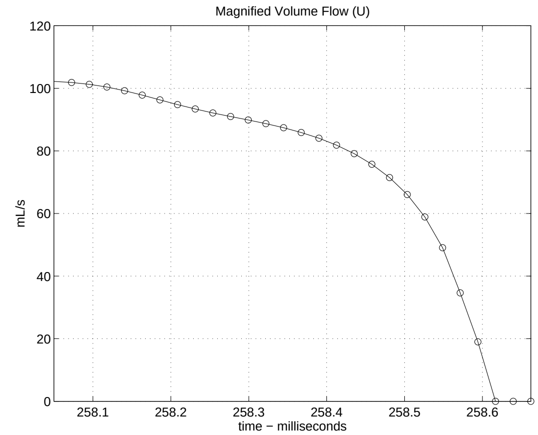

Figure 3: *A magnified view of volume flow showing truncation on a sample boundary.*

is open, the pressure difference across the valve channel and the volume flow through the valve drive the volume flow derivative, and in turn influence the valve position and the pressure on either side of the valve. When the valve is closed, there is no volume flow through the valve, and the pressure on either side of the valve is free to evolve independently. These two behaviours are expressed every cycle as the valve opens and closes in response to an input pressure and the energy reflected from the trachea opening.

The four model variables are simulated by discretizing their corresponding differential equations. The valve motion derivative is expressed in terms of the volume flow and pressures, the pressure derivative as a function of the flow and valve geometry, and the volume flow derivative as a function of the volume flow, pressures, and valve position [2, 3]. The volume flow is the focus in this paper and is given by

$$\frac{dU}{dt} = \frac{2\sqrt{A(t)}}{\rho}(p_0 - p_1) - \frac{U^2}{4A(t)^{\frac{3}{2}}} \tag{1}$$

where $A(t) = ax(t)$ is half the cross-sectional area of the valve.

In simulating the valve, care must be taken in computing the behaviour of volume flow between open and closed states. This is made more difficult by the singularity in (1) as the valve opening approaches zero.

2.1. Model Aliasing

Though the waveforms in Fig. 2 may seem well behaved, they are in fact heavily aliased. In Fig. 7, an example of the model output spectrum is shown for a constant input pressure and an increasing syrinx tension. As the syrinx tension increases, the pitch increases. There is a corresponding decrease in the pitch of the aliased components, creating a clearly visible crosswork pattern.

It is well known that aliasing is caused by switching based on a level threshold in discrete time signals. Thus, as Fig. 2 indicates, aliasing is caused by the truncation of the volume flow $U(t)$ to zero when the valve closes. As illustrated by the magnified plot of volume flow in Fig. 3, $U(t)$ is forced to zero on a sample boundary. Regardless of the value of $U(t)$ predicted by updating (1), air volume flow is still being set to zero when the valve is closed since no air should flow through a closed valve.

Truncating the volume flow on sample boundaries is problematic. Depending on the period of the signal, the clipping may not happen at the correct phase and aliased components will be generated. This is illustrated in Figure 4 which shows a sinusoid and its truncated version along with their respective power spectra. Aliased components appear as peaks at nonharmonic frequencies.

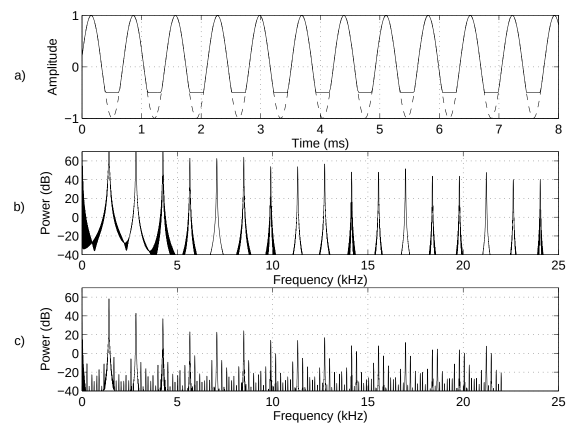

Figure 4: *Figure (a) shows a full (dotted line) and truncated (solid line) version of a sine wave. Plot (b) shows the desired unaliased power spectrum of the solid line waveform in (a) and plot (c) shows the artifacts introduced in the spectrum as a result of abruptly truncating the sinewave.*

3. VOLUME FLOW BEHAVIOUR

Consider the case where half the valve channel area $A(t) = ax(t)$ is sufficiently small that the first term of (1) can be ignored and the differential equation for $U(t)$ can be approximated by

$$\frac{dU}{dt} \approx -\frac{U^2}{4A(t)^{\frac{3}{2}}}, \qquad A(t) \ll 1, \tag{2}$$

which is in the form of a so-called *Bernoulli differential equation* [8]. Though this differential equation is nonlinear in $U(t)$, it may be converted to a linear form by the substitution

$$W(t) = \frac{1}{U(t)}. \tag{3}$$

Writing (2) in terms of $W(t)$ gives the following new differential equation for $U(t)$

$$\frac{dU}{dt} = -\frac{1}{W(t)^2}\frac{dW}{dt}, \tag{4}$$

where

$$\frac{dW}{dt} = \frac{1}{4A(t)^{3/2}}. \tag{5}$$

This equation is easily integrated to solve for volume flow:

$$W(t) = \int^t \frac{d\tau}{4A(\tau)^{3/2}} + C, \tag{6}$$

$$U(t) = \frac{1}{\int^t \frac{d\tau}{4A(\tau)^{3/2}} + C}, \tag{7}$$

where the constant of integration C may be set given knowledge of $U(t)$ at a particular time t_0. Solving for C,

$$U(t) = \frac{U(t_0)}{1 + U(t_0)\int_{t_0}^t \frac{d\tau}{4A(\tau)^{3/2}}}. \tag{8}$$

Note that when the area $A(t)$ is small, the integral in the denominator of (8) is large, and any initial positive value of volume flow is quickly reduced to zero without crossing zero, as would be expected for a closing valve. This observation provides justification for having zero volume flow when the valve area is zero. The small valve area solution to (8) suggests a possible alternative to truncating U when the valve is closed (the original solution to the singularity in (1) that resulted in aliasing). If the valve were slightly leaky, e.g.,

$$\tilde{A}(t) = A(t) + \lambda, \tag{9}$$

for a small leakage area λ, the singularity at zero area would be avoided, and the volume flow behaviour would be relatively unchanged. However, it is not sufficient to use a leaky valve in place of one that is truncated because though this may reduce the slope of $U(t)$ it also introduces the undesirable behaviour of volume flow oscillating about zero.

4. CORRECTED VOLUME FLOW UPDATE

The difficulty with discretizing (1) in the presence of small valve areas is illustrated in Fig. 5. Since, as Fig. 3 indicates, the slope of $U(t)$ is decreasing with decreasing volume flow, predictions of the slope based on (1) tend to overshoot zero volume flow. It would therefore be preferable to use the value of $dU(t)/dt$ as predicted by the small area solution (8) to update the volume flow when $A(t)$ is small.

In order to see how this solution should be incorporated into the volume flow update, consider the value of $U(t)$ at time $t_0 + T$, T being the sampling period. Starting with the small area solution for volume flow (8), but in a more convenient form

$$U(t) = \left[\frac{1}{U(t_0)} + \int_{t_0}^t \frac{1}{4A(\tau)^{3/2}} d\tau\right]^{-1}, \tag{10}$$

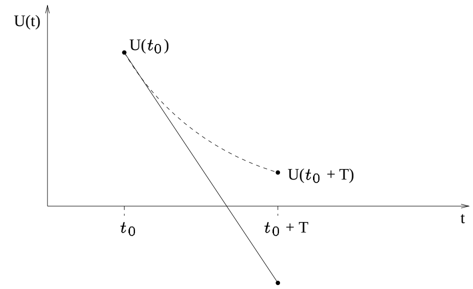

Figure 5: *In the case of a large sampling period T, updating the volume flow using (1) can cause U to overshoot. The dotted line represents the actual value of U.*

the valve channel area is assumed to be constant at $A(t_0)$ during the time interval $[t_0, t_0 + T]$. Substituting into (10)

$$U(t) = \left[\frac{1}{U(t_0)} + \frac{1}{4A(t_0)^{3/2}}(t - t_0)\right]^{-1}, \tag{11}$$

and the volume flow at $t_0 + T$ is

$$U(t_0 + T) = U(t_0)\left[1 + \frac{U(t_0)}{4A(t_0)^{3/2}}T\right]^{-1}. \tag{12}$$

Using the first order backwards difference approximation, the new differential equation for $U(t)$ becomes

$$\begin{aligned} \frac{dU}{dt} &= \frac{U(t_0 + T) - U(t_0)}{T} && (13)\\ &= -\frac{U(t_0)^2}{4A(t_0)^{3/2}} \cdot \left[1 + \frac{U(t_0)}{4A(t_0)^{3/2}}T\right]^{-1}. && (14) \end{aligned}$$

Comparing the form of (14) to (2) note that the Bernoulli terms are identical, save a factor of $\left[1 + U(t_0)/4A(t_0)^{3/2}T\right]^{-1}$. This factor has the effect of reducing the derivative in the presence of small channel areas or large sample periods. Rewriting (14) gives

$$\frac{dU}{dt} = -\frac{U(t_0)^2}{4A(t_0)^{3/2} + U(t_0)T}. \tag{15}$$

Note that in this form the Bernoulli term is similar to that of (2), with a valve having increased area. In other words, it has become a leaky valve whose leakage increases with increasing volume flow.

In addition to creating the desired effect of a gentler slope, this new form for the Bernoulli term (15) solves the numerical instability in (1) by adding a non-zero term to the denominator, allowing $A(t)$ to take on a zero value. It is no longer necessary to manually halt the air flow the moment the valve is closed. Rather, as can be seen in Figure 6, the air flow is now being brought to zero along a more accurate and smooth trajectory.

The final feathered update for volume flow when discretized using the trapezoid rule for numerical integration [1] is given by

$$\begin{aligned} \left.\frac{dU}{dt}\right|_{t_0+T} &= \left.\frac{dU}{dt}\right|_{t_0}\frac{T}{2} + \left(\frac{2\sqrt{A(t_0)}}{\rho}(p_0 - p_1) - \right.\\ &\left.\frac{U(t_0)^2}{4A(t_0)^{\frac{3}{2}} + U(t_0)T}\right)\frac{T}{2}. \qquad (16) \end{aligned}$$

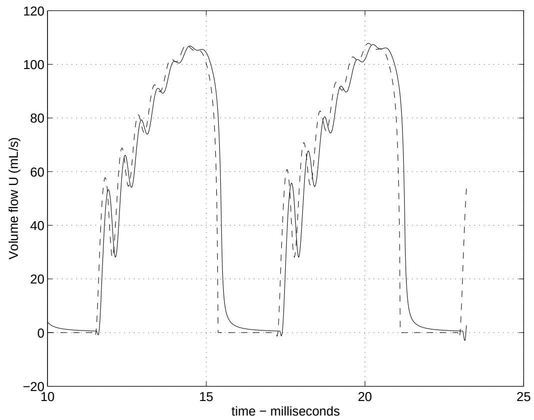

Figure 6: *Volume flow truncated and with "leaky" term added.*

5. CONCLUSIONS

The differential equation (1) describing the behaviour of volume flow (1) can be numerically unstable because of the singularity in the Bernoulli term when the valve closes. Moreover, aliasing is caused by abruptly cutting off flow when the computed flow passes through zero. Both problems are addressed by incorporating the new small-area solution for $U(t)$. The volume flow is now updated in a way which produces smoother transitions between open and close valves. This more accurate and numerically robust solution eliminates the original instability and reduces aliasing, as shown in Fig. 8.

6. REFERENCES

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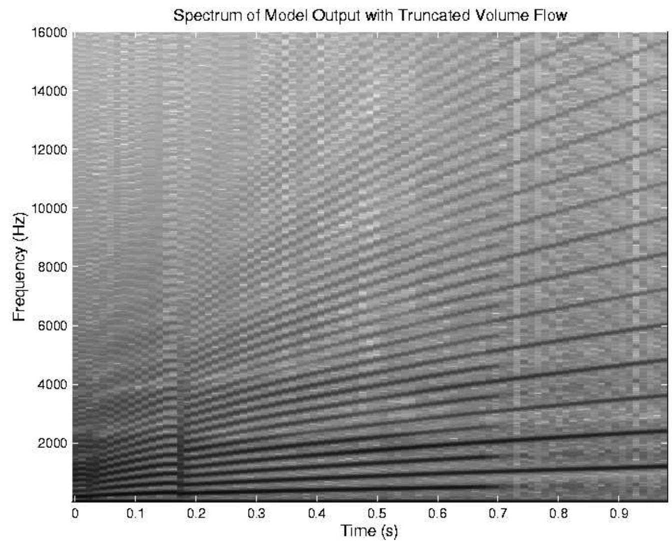

Figure 7: *Model output spectrum with a truncated volume flow.*

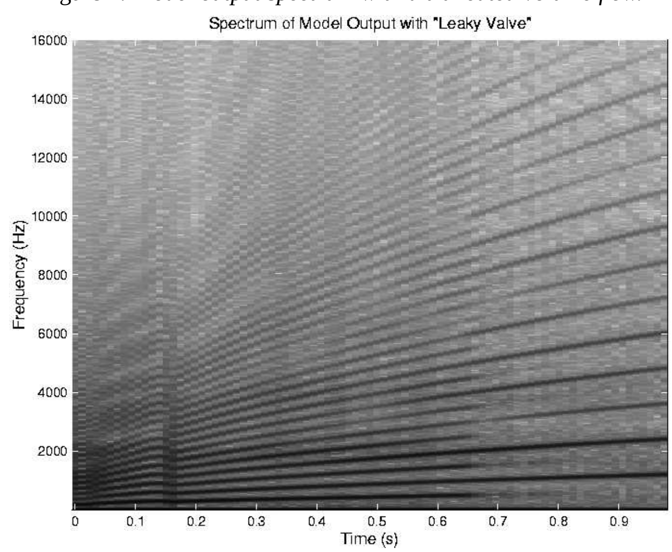

Figure 8: *Model output spectrum with the "leaky" valve.*