Music 270a: Sinusoids

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September 30, 2019
Why Sinusoids are Important

- Sinusoids are fundamental in physics: many systems (e.g. mass-spring) oscillate in a quasi-sinusoidal motion known as *simple harmonic motion*.

- Bandlimited sounds may be viewed as the sum of a finite number of sinusoids having a different amplitude, frequency, and phase.

- The amount of each sinusoidal frequency present in a sound can be viewed by the sound’s *spectrum*.

- The human ear acts as a spectrum analyzer: when sound enters the inner ear, each frequency in the sounds resonates at a position along the *basilar membrane*, where hair cells transmit the vibration to the auditory nerve at the appropriate rate.
Sinusoids

• “Sinusoids” is a collective term referring to both sine and cosine functions.

• A sinusoid is a function having the following form:

\[ x(t) = A \sin(\omega t + \phi) \quad \text{or} \quad x(t) = A \cos(\omega t + \phi), \]

where \( x \) is the quantity which varies over time and

\[ A \triangleq \text{peak amplitude} \]
\[ \omega \triangleq \text{radian frequency (rad/sec) = } 2\pi f \]
\[ f \triangleq \text{frequency (Hz)} \]
\[ t \triangleq \text{time (seconds)} \]
\[ \phi \triangleq \text{initial phase (radians)} \]
\[ \omega t + \phi \triangleq \text{instantaneous phase (radians)} \]

![Figure 1: Sinusoid where \( A = 2, \omega = 2\pi5, \) and \( \phi = \pi/4. \)](image-url)
Amplitude and Magnitude.

• **peak amplitude**: the *nonnegative value* of the waveform’s peak (either positive or negative); often shortened to simply *amplitude*.

• **instantaneous amplitude** of $x$: the value of $x(t)$ (either positive or negative) at time $t$.

• **instantaneous magnitude**: a nonnegative value given by $|x(t)|$; often shortened to simply *magnitude*.
Period

- One cycle of a sinusoid is $2\pi$ radians.
- The period $T$ of a sinusoid is the time (in seconds) it takes to complete one cycle.

![Sinusoid](image)

Figure 2: Sinusoid.

- Since sinusoids are periodic with period $2\pi$, an initial phase of $\phi$ is indistinguishable from an initial phase of $\phi \pm 2\pi$.
- We may therefore restrict the range of $\phi$ so that it does not exceed $2\pi$. Typically we choose the range $-\pi < \phi < \pi$,
  but we may also encounter $0 < \phi < 2\pi$. 

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Phase

- The initial phase $\phi$ (also called phase offset or phase shift), given in radians, tells us the position of the waveform cycle at $t = 0$.

Figure 3: Sine function $\phi = 0$.

Figure 4: Sine function with $\phi = \pi/2$. 

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Frequency

- The frequency $f$ of the waveform is given in cycles per second or Hertz (Hz).
- Frequency is equivalent to the inverse of the period $T$ of the waveform,
  \[ f = \frac{1}{T} \text{ Hz.} \]
- The radian frequency $\omega$, given in radians per second, is equivalent to the frequency in Hertz scaled by $2\pi$, \[ \omega = 2\pi f \text{ (rad/sec)}. \]
- The radian frequency is equal to the time derivative of the instantaneous phase of the sinusoid:
  \[ \omega = \frac{d}{dt}(\omega t + \phi) \]
- Note: if the phase is time varying, the instantaneous frequency is given by
  \[ \frac{d}{dt}[\omega t + \phi(t)] = \omega + \frac{d}{dt}\phi(t). \]
Sine and cosine functions.

- The sine and cosine function are very closely related and can be made equivalent simply by adjusting their initial phase:

\[ \sin \theta = \cos \left( \theta - \frac{\pi}{2} \right) \quad \text{or} \quad \cos \theta = \sin \left( \theta + \frac{\pi}{2} \right). \]

![Diagram showing the phase relationship between cosine (solid blue line) and sine (broken green line) functions.](image)

Figure 5: Phase relationship between cosine (solid blue line) and sine (broken green line) functions.

- In calculus, the sine and cosine functions are derivatives of one other. That is,

\[ \frac{d}{dt} \sin \theta = \cos \theta \quad \text{and} \quad \frac{d}{dt} \cos \theta = -\sin \theta. \]
Time-shifting a signal.

- If a signal can be expressed in the form

\[ x(t) = s(t - t_1), \]

we say \( x(t) \) is a time-shifted version of \( s(t) \).

- Consider the simple function

\[ s(t) = t \quad 0 \leq t \leq 1. \]

- Shifting the function by \( t_1 = 2 \) seconds yields

\[ x(t) = s(t - 2) = t - 2 \quad 0 \leq t - 2 \leq 1 \]
\[ = t - 2 \quad 2 \leq t \leq 3, \]

which is simply \( s(t) \) with its origin shifted to the right, or delayed, by 2 seconds.
• Shifting the function by $t_1 = -1$ seconds yields

$$y(t) = s(t + 1) = t + 1 \quad 0 \leq t + 1 \leq 1$$

$$= t + 1 \quad -1 \leq t \leq 0,$$

which is simply $s(t)$ with its origin shifted to the left, or **advanced in time**, by 1 seconds.

• **Summary:**

  – a **positive** phase indicates a shift to the left on the time axis (advance in time),
  
  – a **negative** phase indicates a shift to the right (a delay in time).
Sinusoidal and Circular Motion

- Consider a vector of length one (1), rotating at a steady speed in a plane, the vector tracing a circle with a radius equal to its length.

![Diagram of a vector rotating along the unit circle.](image)

Figure 7: A vector rotating along the unit circle.

- Each time the vector completes one rotation of the circle, it has completed a cycle of $2\pi$.
- The rate at which the vector completes one cycle is given by its frequency.
- The length of the vector is given by its amplitude (which for simplicity, in this case, is one (1)).
Determining coordinates via Projection

- The x- and y-axis are the horizontal and vertical lines intersecting at the circle’s centre.

![Diagram of vector coordinates](image)

Figure 8: The vector coordinates are determined by projecting onto the x and y-axis.

- **Projecting** the vector onto the x- and y-axes allows us to determine its coordinates in the xy-plane.

- If the vector is rotated in a counterclockwise direction, at angle $\theta$ from the positive x-axis, projecting onto both the x- and y-axes creates right angle triangles.

- Trigonometric identities, with knowledge of $\theta$ and the vector length, will help us determine the coordinates:

$$ y = r \sin(\theta) \quad x = r \cos(\theta). $$

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1Projection can be thought of as a shadow.
Projection of Circular Motion

• Projecting onto the $x$- and $y$-axis gives a sequence of points that resemble a cosine and sine function respectively.

Figure 9: Projecting onto the $x$ and $y$ axis.
In-Phase and Phase-Quadrature Components

• Every sinusoid can be expressed as the sum of a sine and cosine function, or equivalently, an “in-phase” and “phase-quadrature” component.

• Using the trigonometric identity\(^2\)

\[
\sin(a \pm b) = \sin(a) \cos(b) \pm \cos(a) \sin(b),
\]

we see that

\[
A \sin(\omega_0 t + \phi) = A \sin(\phi + \omega_0 t)
= [A \sin \phi] \cos \omega_0 t + [A \cos \phi] \sin \omega_0 t
= B \cos \omega_0 t + C \sin \omega_0 t,
\]

where the amplitude \(A\) is given by

\[
A = \sqrt{B^2 + C^2},
\]

and the phase \(\phi\) is given by

\[
\phi = \tan^{-1} \left( \frac{B}{C} \right).
\]

\(^2\)Also useful is the identity: \(\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b\)
Adding two sinusoids of the same frequency

- Adding two sinusoids of the same frequency but with possibly different amplitudes and phases, produces another sinusoid at that frequency.

![Figure 10: Adding two sinusoids of the same frequency.](image-url)
Vector Addition

- Since one vector represents one sinusoid, to add two sinusoids of the same frequency, we need only perform vector addition.

![Vector Addition Diagram](image)

- Since the vectors have the same frequency, they will rotate as a unit and their sum will have the same frequency.

- The sum vector $U + V$ in Figure 11 also has its own $x$ and $y$ component (from projecting onto the $x$- and $y$-axes) and therefore may have a different amplitude and phase.