Music 270a: Sinusoids

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Why Sinusoids are Important

- Sinusoids are fundamental in physics: many systems (e.g. mass-spring) oscillate in a quasi-sinusoidal motion known as *simple harmonic motion*.
- Bandlimited sounds may be viewed as the sum of a finite number of sinusoids having a different amplitude, frequency, and phase.
- The amount of each sinusoidal frequency present in a sound can be viewed by the sound's *spectrum*.
- The human ear acts as a spectrum analyzer: when sound enters the inner ear, each frequency in the sounds resonates at a position along the *basilar membrane*, where hair cells transmit the vibration to the auditory nerve at the appropriate rate.

Sinusoids

- "Sinusoids" is a collective term referring to both sine and cosine functions.
- A sinusoid is a function having the following form:

$$x(t) = A\sin(\omega t + \phi) \quad \text{or} \quad x(t) = A\cos(\omega t + \phi),$$

where x is the quantity which varies over time and

$$\begin{aligned} A &\triangleq \text{peak amplitude} \\ \omega &\triangleq \text{radian frequency (rad/sec)} = 2\pi f \\ f &\triangleq \text{frequency (Hz)} \\ t &\triangleq \text{time (seconds)} \\ \phi &\triangleq \text{initial phase (radians)} \\ \omega t + \phi &\triangleq \text{instantaneous phase (radians)} \end{aligned}$$

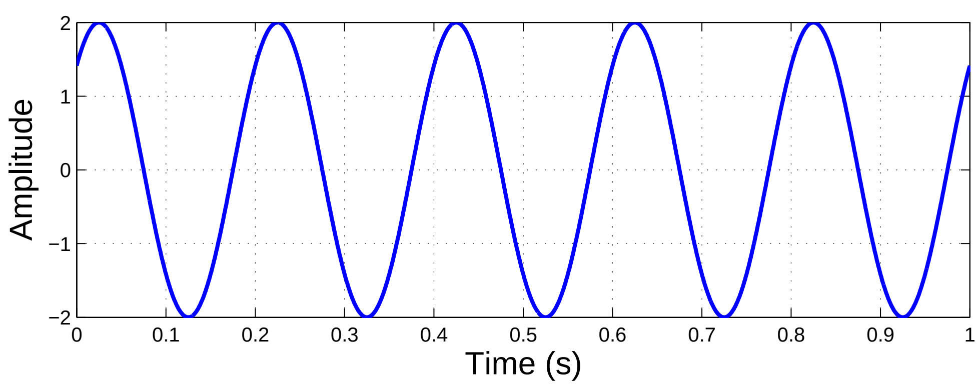

Figure 1: Sinusoid where $A = 2$, $\omega = 2\pi 5$, and $\phi = \pi/4$.

Amplitude and Magnitude.

- **peak amplitude**: the *nonnegative value* of the waveform's peak (either positive or negative); often shortened to simply *amplitude*.
- **instantaneous amplitude** of x: the value of $x(t)$ (either positive or negative) at time t.
- **instantaneous magnitude**: a nonnegative value given by $|x(t)|$; often shortened to simply *magnitude*.

Period

- One **cycle** of a sinusoid is 2π radians.
- The **period** T of a sinusoid is the time (in seconds) it takes to complete one cycle.

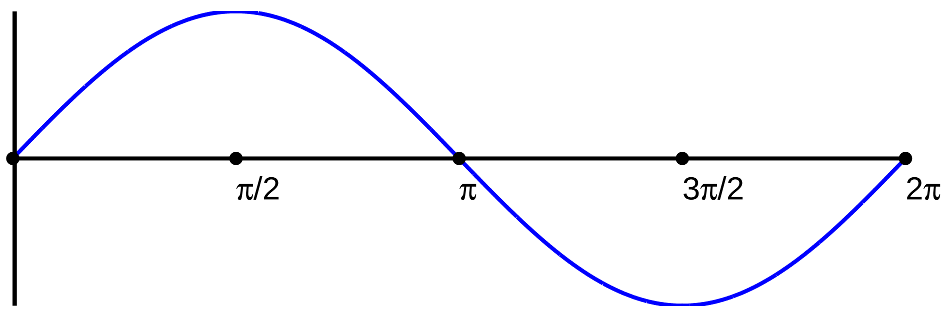

Figure 2: Sinusoid.

- Since sinusoids are periodic with period 2π, an initial phase of ϕ is indistinguishable from an initial phase of $\phi \pm 2\pi$.
- We may therefore restrict the range of ϕ so that it does not exceed 2π. Typically we choose the range

$$-\pi < \phi < \pi,$$

but we many also encounter

$$0 < \phi < 2\pi.$$

Phase

- The **initial phase** ϕ (also called *phase offset* or *phase shift*), given in radians, tells us the position of the waveform cycle at $t = 0$.

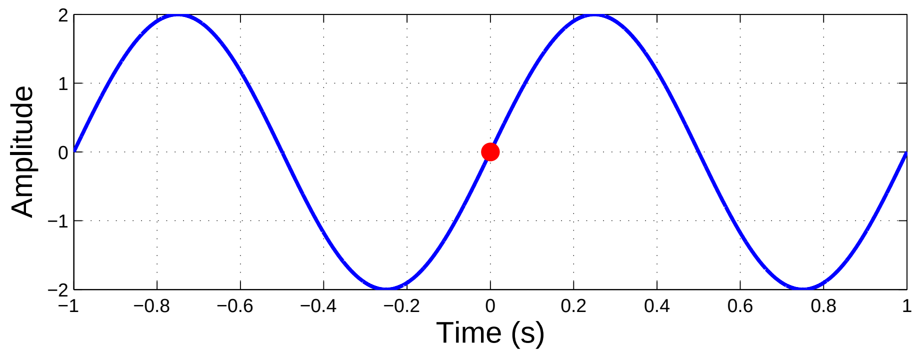

Figure 3: Sine function $\phi = 0$.

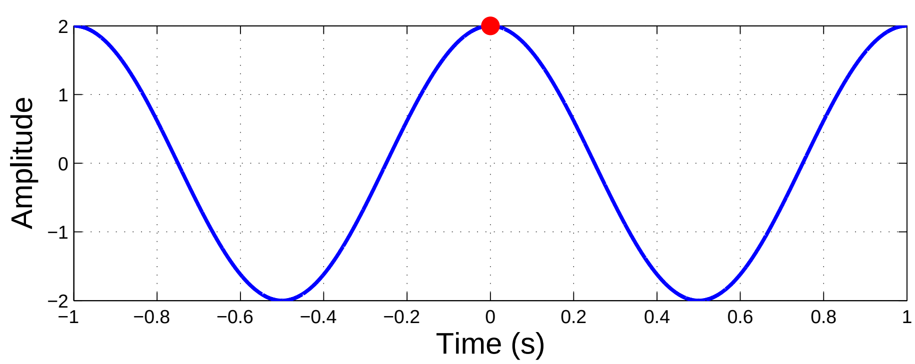

Figure 4: Sine function with $\phi = \pi/2$.

Frequency

- The frequency f of the waveform is given in cycles per second or Hertz (Hz).
- Frequency is equivalent to the inverse of the period T of the waveform,

$$f = 1/T \quad \text{Hz.}$$

- The radian frequency ω, given in radians per second, is equivalent to the frequency in Hertz scaled by 2π,

$$\omega = 2\pi f \quad \text{(rad/sec).}$$

- The radian frequency is equal to the time derivative of the *instantaneous phase* of the sinusoid:

$$\omega = \frac{d}{dt}(\omega t + \phi)$$

- Note: if the phase is time varying, the *instantaneous frequency* is given by

$$\frac{d}{dt}[\omega t + \phi(t)] = \omega + \frac{d}{dt}\phi(t).$$

Sine and cosine functions.

- The sine and cosine function are very closely related and can be made equivalent simply by adjusting their initial phase:

$$\sin\theta = \cos(\theta - \frac{\pi}{2}) \qquad \text{or} \qquad \cos\theta = \sin(\theta + \frac{\pi}{2}).$$

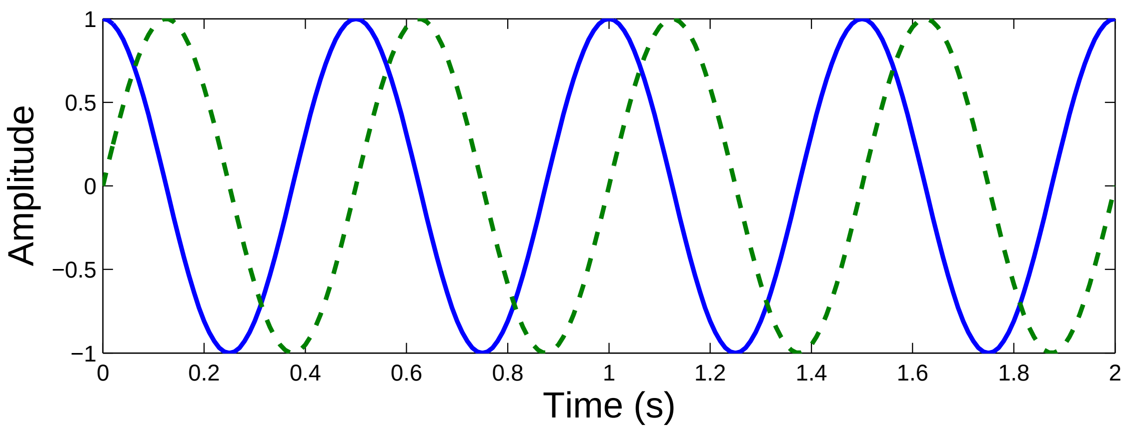

Figure 5: Phase relationship between cosine (solid blue line) and sine (broked green line) functions.

- In calculus, the sine and cosine functions are derivatives of one other. That is,

$$\frac{d\sin\theta}{dt} = \cos\theta \qquad \text{and} \qquad \frac{d\cos\theta}{dt} = -\sin\theta.$$

Time-shifting a signal.

- If a signal can be expressed in the form

$$x(t) = s(t - t_1),$$

we say $x(t)$ is a *time-shifted* version of $s(t)$.

- Consider the simple function

$$s(t) = t \qquad\qquad 0 \leq t \leq 1.$$

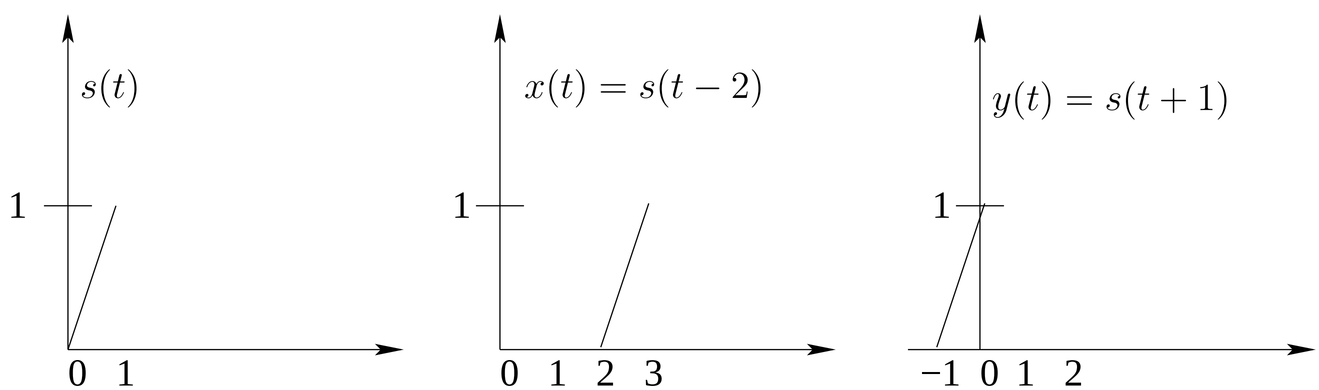

Figure 6: Time-shifting a signal.

- Shifting the function by $t_1 = 2$ seconds yields

$$\begin{aligned} x(t) = s(t-2) &= t-2 \qquad 0 \leq t-2 \leq 1 \\ &= t-2 \qquad 2 \leq t \leq 3, \end{aligned}$$

which is simply $s(t)$ with its origin shifted to the right, or **delayed**, by 2 seconds.

- Shifting the function by $t_1 = -1$ seconds yields

$$\begin{aligned} y(t) = s(t+1) &= t+1 \qquad 0 \leq t+1 \leq 1 \\ &= t+1 \qquad -1 \leq t \leq 0, \end{aligned}$$

  which is simply $s(t)$ with its origin shifted to the left, or **advanced in time**, by 1 seconds.

- Summary:
  - a **positive** phase indicates a shift to the left on the time axis (advance in time),
  - a **negative** phase indicates a shift to the right (a delay in time).

Sinusoidal and Circular Motion

- Consider a vector of length one (1), rotating at a steady speed in a plane, the vector tracing a circle with a radius equal to its length.

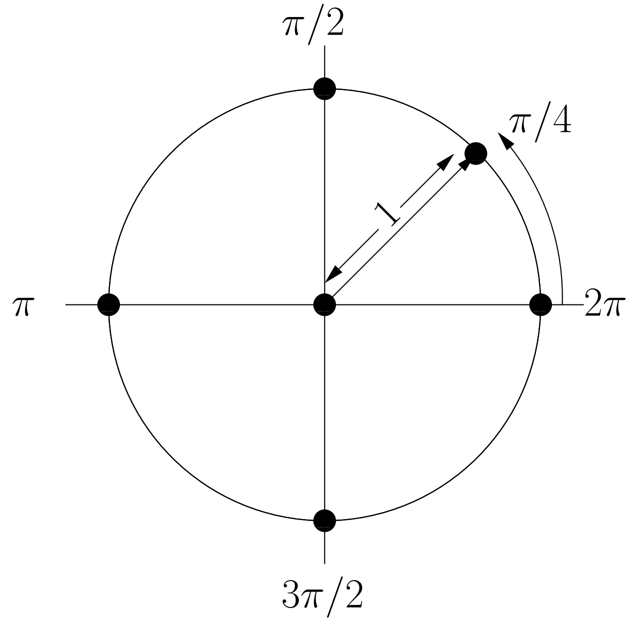

Figure 7: A vector rotating along the unit circle.

- Each time the vector completes one rotation of the circle, it has completed a cycle of 2π.
- The rate at which the vector completes one cycle is given by its frequency.
- The length of the vector is given by its amplitude (which for simplicity, in this case, is one (1)).

Determining coordinates via Projection

- The x- and y-axis are the horizontal and vertical lines intersecting at the circle's centre.

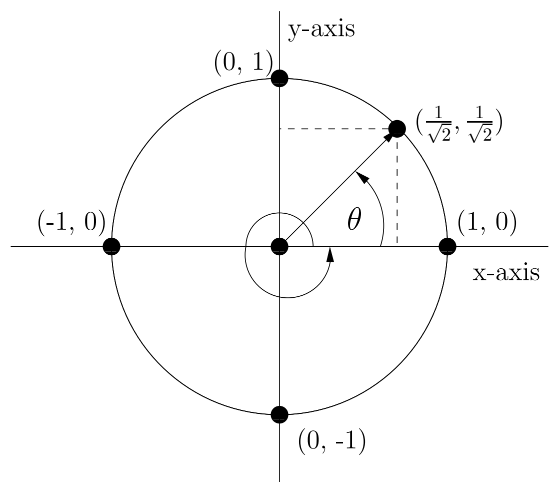

Figure 8: The vector coordinates are determined by projecting onto the x and y-axis.

- *Projecting*[1] the vector onto the x- and y-axes allows us to determine its coordinates in the xy-plane.
- If the vector is rotated in a counterclockwise direction, at angle θ from the positive x-axis, projecting onto both the x- and y-axes creates right angle triangles.
- Trigonometric identities, with knowledge of θ and the vector length, will help us determine the coordinates:

$$y = r \sin(\theta) \qquad x = r \cos(\theta).$$

[1]Projection can be thought of as *a shadow*.

Projection of Circular Motion

- Projecting onto the x- and y-axis gives a sequence of points that resemble a cosine and sine function respectively.

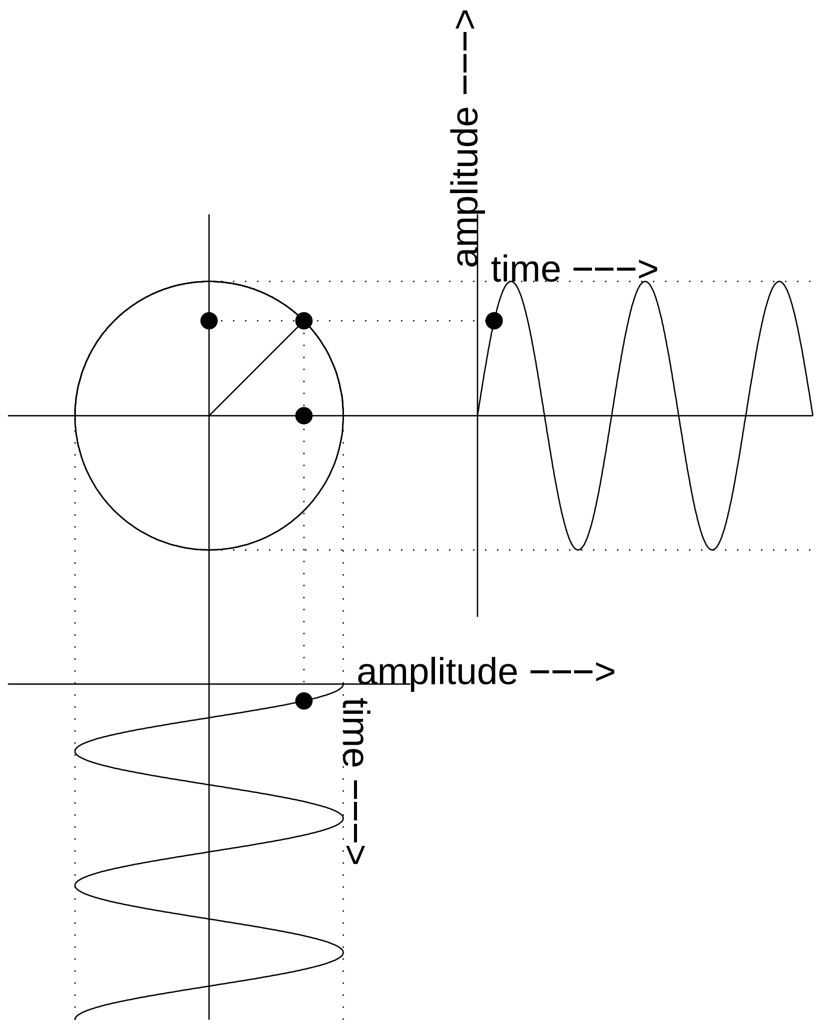

Figure 9: Projecting onto the x and y axis.

In-Phase and Phase-Quadrature Components

- Every sinusoid can be expressed as **the sum of a sine and cosine function**, or equivalently, an "in-phase" and "phase-quadrature" component.
- Using the trigonometric identity[2]

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b),$$

we see that

$$\begin{aligned} A\sin(\omega_0 t + \phi) &= A\sin(\phi + \omega_0 t) \\ &= [A\sin\phi]\cos\omega_0 t + [A\cos\phi]\sin\omega_0 t \\ &= B\cos\omega_0 t + C\sin\omega_0 t, \end{aligned}$$

where the amplitude A is given by

$$A = \sqrt{B^2 + C^2},$$

and the phase ϕ is given by

$$\phi = \tan^{-1}\left(\frac{B}{C}\right).$$

[2] Also useful is the identity: $\cos(a \pm b) = \cos a \cos b \mp \cos a \cos b$

Adding two sinusoids of the same frequency

- Adding two sinusoids of the same frequency but with possibly **different amplitudes and phases**, produces another sinusoid at that frequency.

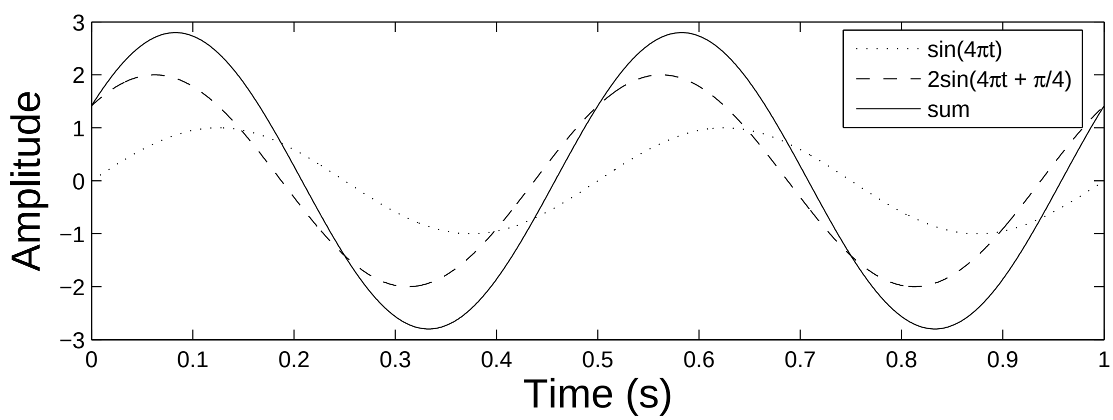

Figure 10: Adding two sinusoids of the same frequency.

Vector Addition

- Since one vector represents one sinusoid, to add two sinusoids of the same frequency, we need only perform vector addition.

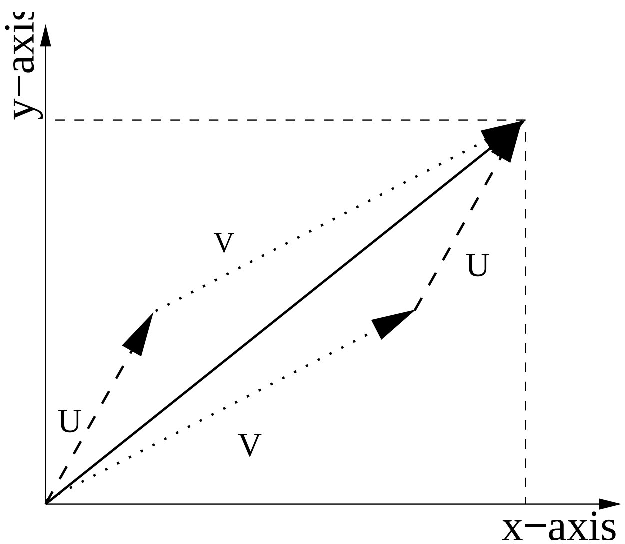

Figure 11: Adding sinusoids using vector addition.

- Since the vectors have the same frequency, they will rotate as a unit and their sum will have the same frequency.
- The sum vector U + V in Figure 11 also has its own x and y component (from projecting onto the x- and y-axes) and therefore may have a different amplitude and phase.