Vibrations of a String

• When a string is excited (struck, plucked, bowed or otherwise disturbed) the displacement wave on the string is
  – sum of right and left traveling waves,
  – described by D’Alembert’s traveling wave solution to the wave equation:
    \[ y = f_1(ct - x) + f_2(ct + x), \]
  where \( c \) is the wave speed and
    \[ f_1(ct - x) \triangleq \text{a right traveling wave} \]
    \[ f_2(ct + x) \triangleq \text{a left traveling wave}. \]

Standing Waves

• Strings in musical instruments are usually fixed at both ends and thus the displacement wave is inverted upon reflection:

  – see reflection from a fixed boundary

  – Recall, reflection causes destructive and constructive interference, leading to standing waves:
    * See animation: standing waves created from a fixed boundary
  – The fundamental mode of oscillation, is determined by the shortest node-antinode pattern.

String Fixed at Both Ends

• The guitar string is fixed at bridge and nut:
  – standing wave pattern is constrained to have nodes at both ends:
  – first 3 modes on a string fixed at both ends:

    – for string length \( L \), each mode of oscillation \( n \) has wavelength:
      \[ \lambda_n = \frac{2L}{n} \]
      and frequency:
      \[ f_n = \frac{c}{\lambda_n} = \frac{cn}{2L} = nf_1, \]
      where \( f_1 \) is the first mode of oscillation corresponding to the fundamental frequency.
The Plucked String

• When a string is *plucked* it is displaced transversely at a point along its length.
  – plucking a string introduces an initial displacement and thus potential energy.

• When the string is released, the resulting vibration is a combination of the normal modes of vibration.

• Notice a string plucked in the middle looks like a triangle wave—why?

Plucking in the Middle

• When plucked in the middle, the string displacement prevents a node from forming at that position:
  – all modes of oscillation having a node in that position cannot occur;

  – the resulting waveform has only odd harmonics (like a triangle wave);

  – positive displacement results in alternating phase between adjacent modes.

Plucking Position

• The shape of the string before its release defines which harmonics will be present in the resulting motion.
  – a string plucked at $1/n^{th}$ the string length (from one end) will not have energy at multiples of the $n^{th}$ harmonic.
  – the strength of the $n^{th}$ mode is inversely proportional to the square of the mode number.
  – example: the spectrum of a string plucked at $1/5^{th}$ the string length:

Motion of Initial Pluck

• The motion of a string plucked one-fifth at one-fifth of the string length:

  – The motion can be seen as
    – the sum of two pulses (left and right traveling waves) moving in opposite directions (dashed line);
    – consisting of two bends, one moving clockwise and the other counterclockwise around a parallelogram.

  – See traveling2.m (shown in class).
Delay Line

• **Delay line**: models *acoustic propagation delay*.

\[ \begin{align*}
  x(n) & \quad \text{M sample delay} \quad y(n)
\end{align*} \]

– doesn’t change waveform shape—just delays it for a certain period of time (number of samples).
– for input \( x(n) \), output is \( y(n) = x(n - M) \).
– may be used to model round-trip propagation time along the length of the string.

![Figure 1: The M-sample delay line.](image)

Calculating Delay

• Given a wave speed of \( c \) and a sampling period of \( T_s = 1/f_s \)
  – distance the wave to propagate in one time sample is \( X = cT_s \),
  – number of delay samples corresponding to * \( L \) meters:
    \[ M = \frac{L}{X} \],
  * sounding frequency \( f_0 \):
    \[ M = \frac{f_s}{f_0} \].

Karplus-Strong Model

• Taking the output of the delay line, multiplying it by frequency-dependent losses (low-pass filter), and feeding it back into the input gives you the **Karplus-Strong** simple string model.

\[ \begin{align*}
  y(n) & \quad \text{N sample delay} \quad y(n - N) \quad \text{LPF} \quad y(n)
\end{align*} \]

![Figure 2: A very simple model of a rigidly terminated string with low-pass filter (LPF).](image)

• In Pd, you can use the `vdelay` object to create a specified `sdelay` in seconds.

• See [karplusstrong.pd](karplusstrong.pd) NOTE: this does not play in tune because of the delay introduced by the feedback!

• This can be shown to be equivalent to a simple 1-D waveguide model.

Digital Waveguide

• Alternatively, one may use delaylines to model the right and left traveling waves.

• As per D’Alembert’s solution to the wave equation, physical variables (such as string displacement) are obtained by summing traveling-wave components.

\[ \begin{align*}
  & \quad \text{N sample delay} \\
  + & \quad \text{Physical Signal} \\
  & \quad \text{N sample delay}
\end{align*} \]

• To determine the value at any physical point, extract a physical signal from a digital waveguide using delay-line taps and add them together.
Plucked String Waveguide Model

- In this string simulator, there is a loop of delay containing
  \[ N = 2L/X = f_s/f_1 \] samples
  where \( f_1 \) is the sounding frequency of the string of length \( L \).

\[ y(n)^+ \]
\[ \text{N/2 sample delay} \]
\[ y(n-\text{N/2})^+ \]
\[ \text{Physical Signal} \]
\[ \text{Bridge (rigid termination)} \]
\[ \text{y(n)^-} \]
\[ \text{N/2 sample delay} \]
\[ y(n-\text{N/2})^- \]

Resonator

- **Resonance**: when one vibrating system drives another to oscillate with greater amplitude at a specific frequency.
- Resonators are vibrating systems that can experience resonance.
- See [Exploratorium video on resonance](#).
- The modes of oscillation are the resonant frequencies of the resonator
  - the pendulum is a resonator with 1 resonant frequency;
  - the string has multiple resonances, each harmonics of a fundamental frequency.

String Resonance

- For the string to resonate, oscillations must be reinforced.
- This occurs when phase of the driving sinusoid, after traveling round trip the length of the string, is equal to the initial phase.
- For a round-trip propagation delay of 2 seconds:

\[ \text{String output (green) when driving by periodic input (blue)} \]

Vibrations of a Bowed String

- When the bow is drawn across the string of a violin, the string appears to vibrate smoothly between two curved boundaries.
- In reality, the string is nearly straight with a sharp bend that races along the string following a curved path.
- Because of its speed, our eye sees only the curved envelope.
- See [violin string in slow motion](#).
Stick-Slip Bowed-String Mechanism

- The beginning and the end of the slip are triggered by the arrival of the bend:
  -- as the bow moves upward along the string, the string is carried along by the bow (stick);
  -- when the bend returns to the bow from the far (fixed) end, the tension in the string causes it to detach from the bow (slip);
  -- the string slips from the bow, until the bend returns from the shorter (fixed) end;
  -- since string is going at the same speed and direction as the bow, bow catches the string, and the cycle repeats.

- See animation: stick-slip

Friction and Stick-Slip

- The force of friction between the bow and string is less during the "slip" than during the "stick".
- Displacement of the string and bow at contact point:

- Stick: up to a, and from c to i, the string moves at the constant speed of the bow.
- Slip: from a to c, the string makes a rapid return until caught up by a different point on the bow.
- The time required for one round-trip propagation depends only on string length and wave speed
  -- frequency of vibration remains the same under widely varying bowing conditions.

Bowing Conditions

- Bowing is limited by the conditions under which the bend can trigger the beginning and end of the "slip".
- For each position of the bow along the string, there is a maximum and minimum bowing force.
- The closer the bow is to the bridge, the less leeway between minimum and maximum bowing force.

String Instrument Body

- The small diameter strings displace very little air as they vibrate—they radiate little sound on their own.
- Sound quality (and playability) of string instruments is dependent also on the behaviour of their body (guitar, violin, etc.).
- The vibrating strings set the body into vibration, resulting in considerably more strength in the radiated sound.
• The violin consists of strings effectively terminated by a **bridge** and **nut** and a **body**:
  
  – top and back plate, and ribs;
  – two **f-holes** (top plate).
  – **bass bar** (top plate) under one foot of bridge;
  – **sound post** under other foot of bridge, extending from top to back plate;

• Sound is mainly from the top plate, but influenced by bridge driving force and body resonances.

**Violin Bridge**

• The vibrating string exerts a sideways force on the bridge, which in turn transmits this periodic driving force to the top plate.

• The bridge has strong resonances around 3000 and 6000 Hz:
  
  – lower resonances due to rocking-bending motion,
  – upper resonances due to a symmetrical up-and-down motion.

• Shaping the bridge (or attaching a mute) is a convenient way to alter the violin’s frequency response.

**Violin Body and Chladni Patterns**

• Recall, **nodes** were the positions in a standing wave where there was no vibration due to the complete cancelation of summed right and left traveling waves.

• A plate or membrane (2-D) is divided into vibrating regions by **nodal lines** (where there is zero vibration).

• Different geometries have different patterns of nodal lines:
  
  – see [Chladni patterns](#).

• For a rectangular plate/membrane, modes are identified by two numbers \((n, m)\):
  
  – \(n\): number of nodes running parallel to long axis,
  – \(m\): number in the perpendicular direction.
  
  – see animation: [rectangular modes](#).

• For a circular plate/membrane, modes \((n, m)\) refer to:
  
  – \(n\): number of nodes that are diameters (straight lines),
  – \(m\): number of circular nodes.
  
  – see animation: [circular modes](#).

**Vibrations of the Violin Body**

• Chladni patterns become increasingly complex for more complex geometries.

• The first 7 modes \(a\) for violin plate:
Tuning Top and Back Plates

- The top and back plates are carved from blocks of wood, thicker at the center.
- While shaping to plates, the violin maker tests them by listening to tap tones.
- See Intro to Tap Tuning
- Modes for the belly are complicated by the presence of the f-holes and the bass bar:
  - f-holes make the plate more flexible along the short axis
  - the bass bar make the plate less flexible along the long axis.
- See Violin modes

Guitar

- In the guitar, the strings are also fastened directly to the bridge and are coupled to the top plate.
- Both top and back plates are braced, one of the critical parameters in guitar design.
- Braces strengthen the fragile plate (adds stiffness across the grain) and also transmit vibrations of the bridge to the various parts of the body.

Chladni patterns of a guitar top plate

- from University of New South Wales

Guitar as a Vibrating System

- Like the violin, the guitar can be considered a system of coupled oscillators.
- Sound is radiated efficiently by the vibrating plates and through the sound hole:
  - low frequencies: the top plate transmits energy to the back plate and the sound hole via the air cavity
  - high frequencies: most of the sound is radiated by the top plate.
- See modal analysis of an acoustic guitar
Electric Guitar

- Electric guitars use magnetic pickups in which the signal of the vibrating string is transformed into an electrical signal.
- May have both solid or hollow bodies, but the body has less influence on the tone than with its acoustic counterpart.
- Solid body:
  - heavier,
  - less susceptible to acoustic feedback from loudspeaker to guitar,
  - strings vibrate slightly longer (less impedance matching with surrounding air).

Pickup Placement

- Most electric guitars have 2-3 pickups for each string.

![Figure 4: Arrangement of multiple pickups to sample various modes of vibration of the string.](image)

- Pickups located at different points along the string sample different strengths of the various harmonics.
  - the front pickup (nearest the fretboard) generates the strongest signal at the fundamental frequency.
  - the rear pickup (nearest the bridge) is most sensitive to higher harmonics.
- Switches or individual gain controls allow the guitarist to mix together the signals from the pickups as desired.

Fret Placement

- Recall, a semitone corresponds to a frequency ratio of $2^{\frac{1}{12}} \approx 1.0595$, which is also near the ratio 18:17.
- This has led to the rule of eighteen: each fret should be placed $1/18$ of the remaining distance to the bridge.

![Figure 5: Fret placement according to the rule of eighteen, $x = d/18$.](image)

- The difference between $2^{\frac{1}{12}}$ and the ratio 18:17 is an error of about 0.06%, and thus each semitone will be slightly flat with the rule of eighteen.
- This detuning is cumulative and becomes audible by the twelfth fret.
- For best tuning, 17.817 should be used in place of 18.

String Compensation

- Another design issue is that pressing a string down against a fret increases its tension slightly:
  - thus fretted notes tend to be sharp compared to open notes;
  - the greater the clearance between the strings and frets, the greater this sharpening.
- To compensate, the distance from nut to bridge is made slightly greater than the length used to determine the fret spacings:
  - 1-5 mm on acoustic guitars
  - can be several centimeters on an electric bass
- Amount of needed compensation depends on
  - clearance between strings and fret: a guitar with higher action (larger clearance) will require more compensation;
  - string type: bass strings require more compensation than treble; steel strings more than nylon.
• For some electric guitars, compensation is adjustable for each individual string.