Primary Mechanical Vibrators

- With a model of a wind instrument body, a model of the mechanical "primary" resonator is needed for a complete instrument.
- Many sounds are produced by coupling the mechanical vibrations of a source to the resonance of an acoustic tube:
  - In vocal systems, air pressure from the lungs controls the oscillation of a membrane (vocal fold), creating a variable constriction through which air flows.
  - Similarly, blowing into the mouthpiece of a clarinet will cause the reed to vibrate, narrowing and widening the airflow aperture to the bore.

Mass-spring System

- Before modeling this mechanical vibrating system, let's review some acoustics, and mechanical vibration.
- The simplest oscillator is the mass-spring system:
  - a horizontal spring fixed at one end with a mass connected to the other.

![Figure 1: An ideal mass-spring system.](image)

- The motion of an object can be describe in terms of its
  1. displacement $x(t)$
  2. velocity $v(t) = \frac{dx}{dt}$
  3. acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Equation of Motion

- The force on an object having mass $m$ and acceleration $a$ may be determined using Newton's second law of motion
  $$F = ma.$$  
- There is an elastic force restoring the mass to its equilibrium position, given by Hooke's law
  $$F = -Kx,$$
  where $K$ is a constant describing the stiffness of the spring.
- The spring force is equal and opposite to the force due to acceleration, yielding the equation of motion:
  $$m\frac{d^2x}{dt^2} = -Kx.$$
Solution to Equation of Motion

- The solution to
  \[ m\frac{d^2x}{dt^2} = -Kx, \]
  is a function proportional to its second derivative.
- This condition is met by the sinusoid:
  \[ x = A \cos(\omega_0 t + \phi) \]
  \[ \frac{dx}{dt} = -\omega_0 A \sin(\omega_0 t + \phi) \]
  \[ \frac{d^2x}{dt^2} = -\omega_0^2 A \cos(\omega_0 t + \phi) \]
- Substituting into the equation of motion yields:
  \[ m\frac{d^2x}{dt^2} = -Kx, \]
  showing that the natural frequency of vibration is
  \[ \omega_0 = \sqrt{\frac{K}{m}}. \]

Motion of the Mass Spring

- We can make the following inferences from the oscillatory motion (displacement) of the mass-spring system written as:
  \[ x(t) = A \cos(\omega_0 t) \]
  - Energy is conserved, the oscillation never decays.
  - At the peaks:
    * the spring is maximally compressed or stretched;
    * the mass is momentarily stopped as it is changing direction;
  - At zero crossings,
    * the spring is momentarily relaxed (it is neither compressed nor stressed), and thus holds no PE
    * all energy is in the form of KE.
  - Since energy is conserved, the KE at zero crossings is exactly the amount needed to stretch the spring to displacement $-A$ or compress it to $+A$.

PE and KE in the mass-spring oscillator

- All vibrating systems consist of this interplay between an energy storing component and an energy carrying ("massy") component.
- The potential energy PE of the ideal mass-spring system is equal to the work done\(^1\) stretching or compressing the spring:
  \[ PE = \frac{1}{2} Kx^2, \]
  \[ = \frac{1}{2} KA^2 \cos^2(\omega_0 t + \phi). \]
- The kinetic energy KE in the system is given by the motion of mass:
  \[ KE = \frac{1}{2} m v^2, \]
  \[ = \frac{1}{2} m \omega_0^2 A^2 \sin^2(\omega_0 t + \phi) \]
  \[ = \frac{1}{2} KA^2 \sin^2(\omega_0 t + \phi). \]

Conservation of Energy

- The total energy of the ideal mass-spring system is constant:
  \[ E = PE + KE \]
  \[ = \frac{1}{2} KA^2 \left( \sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi) \right) \]
  \[ = \frac{1}{2} KA^2 \]

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\(^1\)Work is the product of the average force and the distance moved in the direction of the force.
Other Simple Oscillators—Pendulum

- Besides the mass-spring, there are other examples of simple harmonic motion.
- A pendulum: a mass $m$ attached to a string of length $l$ vibrates in simple harmonic motion, provided $x \ll l$.

![Figure 3: A simple pendulum.](image)

- The frequency of vibration is
  \[ f = \frac{1}{2\pi} \sqrt{\frac{g}{l}}, \]
  where $g$ is the acceleration due to gravity.
- According to this equation, would changing the mass change the frequency?

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Other Simple Oscillators—Helmholtz Resonator

- A Helmholtz Resonator: The mass of air in the neck serves as a piston, and the larger volume as the spring.

![Figure 3: A simple pendulum.](image)

- The resonant frequency of the Helmholtz resonator is given by
  \[ f_0 = \frac{c}{2\pi} \sqrt{\frac{a}{VL}}, \]
  where $a$ is the area of the neck, $V$ is the volume, $L$ is the length of the neck and $c$ is the speed of sound.

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Damped Vibration

- In a real system, mechanical energy is lost due to friction and other mechanisms causing damping.
- Unless energy is reintroduced into the system, the amplitude of the vibrations with decrease with time.
- A change in peak amplitude with time is called an envelope, and if the envelope decreases, the vibrating system is said to be damped.

![Figure 4: A damped sinusoid.](image)

Mass-spring-damper system

- Damping of an oscillating system corresponds to a loss of energy or equivalently, a decrease in the amplitude of vibration.

![Figure 5: A mass-spring-damper system.](image)

- The damper is a mechanical resistance (or viscosity) and introduces a drag force $F_r$ typically proportional to velocity,
  \[ F_r = -Rv = -R \frac{dx}{dt}, \]
  where $R$ is the mechanical resistance.
Damped Equation of Motion

- The equation of motion for the damped system is obtained by adding the drag force into the equation of motion:
  \[ m \frac{d^2x}{dt^2} + R \frac{dx}{dt} + Kx = 0, \]
  or alternatively
  \[ \frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^2_0 x = 0, \]
  where \( \alpha = \frac{R}{2m} \) and \( \omega^2_0 = \frac{K}{m} \).

- The damping in a system is often measured by the quantity \( \tau \), which is the time for the amplitude to decrease to \( \frac{1}{e} \):
  \[ \tau = \frac{1}{\alpha} = \frac{2m}{R}. \]

The Solution to the Damped Vibrator

- The solution to the system equation
  \[ \frac{dx^2}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^2_0 x = 0 \]
  has the form
  \[ x = A(t) \cos(\omega_d t + \phi), \]
  where \( A(t) \) is the amplitude envelope
  \[ A(t) = e^{-t/\tau} = e^{-\alpha t}, \]
  and the natural frequency \( \omega_d \)
  \[ \omega_d = \sqrt{\omega^2_0 - \alpha^2}, \]
  is lower than that of the ideal mass-spring system
  \[ \omega_0 = \sqrt{\frac{K}{m}}. \]

- Peak \( A \) and \( \phi \) are determined by the initial displacement and velocity.

Systems with Several Masses

- When there is a single mass, its motion has only one degree of freedom and one natural mode of vibration.

- Consider the system having 2 masses and 3 springs:

  ![2 mass 3 spring system](image)

  Figure 6: 2 mass 3 spring system.

- The system will have two “normal” independent modes of vibration:
  1. one in which masses move in the same direction, with frequency
     \[ f_1 = \frac{1}{2\pi} \sqrt{\frac{K}{m}} \]
     (assuming equal masses and springs).
  2. one in which masses move in different directions with frequency
     \[ f_2 = \frac{1}{2\pi} \sqrt{\frac{3K}{m}} \]

System Equations for Two Spring-Coupled Masses

- The extensions of the left, middle and right springs are \( x_1, x_2 - x_1, \) and \( -x_2, \) respectively.

- When a spring is extended by \( x \), the mass attached to the
  - left experiences a positive horizontal restoring force \( F_r = Kx \);
  - right experiences an equal and opposite force \( F_r = -Kx \),
  where \( K \) is the spring constant.

- The equation of motion for the displacement of the first mass:
  \[ m\ddot{x}_1 = -Kx_1 + K(x_2 - x_1), \]
  and the second mass,
  \[ m\ddot{x}_2 = -K(x_2 - x_1) + K(-x_2). \]
Additional Modes

• When modes are independent, the system can vibrate in one mode with minimal excitation of another.
• Unless constrained to one-dimension, the masses can also move transversely (at right-angles to the springs).
• An additional mass adds an additional mode of vibration.
• An N-mass system has N modes per degree of freedom.
• As N gets very large, it becomes convenient to view the system as a continuous string with a uniform mass density and tension.

Forced (Driven) Vibration

• Getting the system to vibrate at any single mode of vibration requires exciting, or driving the system at the desired mode.
• See Dan Russell’s site: The forced harmonic oscillator
• When a simple harmonic oscillator is driven by an external force $F(t)$, the equation of motion becomes $m\ddot{x} + R\dot{x} + Kx = F(t)$.
• The driving force may have harmonic time dependence, it may be impulsive, or it can be a random function of time (noise).

Phase of Driven Vibration

• The force driving an oscillation can be illustrated by holding a slinky in the vertical direction.
• Move the hand up and down slowly:
  – At low $f_h$, both the hand and the mass move in the same direction, and the spring hardly stretches at all.
  – Increasing $f_h$ makes it harder to move mass, and it lags behind the driving force.
  – At resonance $f_h = f_0$: the mass is 1/4 cycle behind the hand.
  – The amplitude is at its maximum.
• The higher the $Q$ (quality factor), of the vibrating system, the more abrupt the transition in phase.

Resonance

• At resonance, there is maximum transfer of energy between the hand and the mass-spring system.
• Plotting amplitude $A$ with respect to $f_h$, shows a curve that is almost symmetrical about its peak $A_{max}$, i.e., it has a bandwidth measured a distance down from $A_{max}$.
• Both the peak $A_{max}$, the bandwidth $\Delta f$ depend on the damping in the system:
  1. heavily damped: $\Delta f$ is large and $A_{max}$ is small.
  2. little damping: $\Delta f$ is small and $A_{max}$ is large, creating a sharp resonance.
Quality Factor and Bandwidth

- The bandwidth of the resonance is typically described using the quantity \( Q = \frac{f_0}{\Delta f} \), for quality factor.
- A high-\( Q \) has a sharp resonance, and a low-\( Q \) has a broad resonance curve.
- For a vibrator set into motion and left to vibrate freely, its decay time is proportional to the \( Q \) of its resonance.
- In terms of our mechanical system, given the resistance \( \alpha = \frac{R}{2m} \), the quality factor is:
  \[
  Q = \frac{\omega_0}{2\alpha}.
  \]

How to Discretize?

- The one-sided Laplace transform of a signal \( x(t) \) is defined by
  \[
  X(s) = \mathcal{L}\{x\} = \int_0^\infty x(t) e^{-st} \, dt
  \]
  where \( t \) is real and \( s = \sigma + j\omega \) is a complex variable.
- The differentiation theorem for Laplace transforms states that
  \[
  \frac{dx(t)}{dt} \leftrightarrow sX(s)
  \]
  where \( x(t) \) is any differentiable function that approaches zero as \( t \) goes to infinity.
- The transfer function of an ideal differentiator is \( H(s) = s \), which can be viewed as the Laplace transform of the operator \( d/dt \).
- Given the equation of motion
  \[
  m\ddot{x} + R\dot{x} + Kx = F(t),
  \]
  the Laplace Transform is
  \[
  s^2X(s) + 2\alpha sX(s) + \omega_0^2 X(s) = F(s).
  \]

Finite Difference

- The finite difference approximation (FDA) amounts to replacing derivatives by finite differences, or
  \[
  \frac{dx(t)}{dt} \approx \lim_{\delta \to 0} \frac{x(t) - x(t - \delta)}{\delta} \approx \frac{x(nT) - x[(n-1)T]}{T}.
  \]
- The z transform of the first-order difference operator is \( (1 - z^{-1})/T \). Thus, in the frequency domain, the finite-difference approximation may be performed by making the substitution
  \[
  s \rightarrow \frac{1 - z^{-1}}{T}
  \]
- The first-order difference is first-order error accurate in \( T \). Better performance can be obtained using the bilinear transform, defined by the substitution
  \[
  s \rightarrow c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)
  \]
  where \( c = \frac{2}{T} \).

FDA of Equation of Motion
Pressure Controlled Valves

• Returning now to our model of wind instruments....
• Blowing into an instrument mouthpiece creates a pressure difference across the surface of the reed.
• When the reed oscillates, it creates and alternating opening and closure to the bore, allowing airflow entry during the open phase and cutting it off during the closed phase.
• The effect, is often seen as a periodic train of pressure pulses into the bore.
• Sound sources of this kind are referred to as pressure-controlled valves and they have been simulated in various ways to create musical synthesis models and vocal systems.

Classifying Pressure-Controlled Valves

• The method for simulating the reed typically depends on whether an additional upstream or downstream pressure causes the corresponding side of the valve to open or close further.
• As per Fletcher, the couplet \((\sigma_1, \sigma_2)\) may be used to describe the upstream and downstream valve behaviour, respectively.
• A \(\sigma_n\) value of
  - +1 indicates an opening of the valve,
  - -1 indicates a closing of the value,
on side \(n\) in response to a pressure increase.

Three simple configurations of PC valves

1. \((-+): the valve is blown closed (as in woodwind instruments or reed-pipes of the pipe organ).
2. \((+-): the valve is blown open (as in the simple lip-reed models for brass instruments, the human larynx, harmonicas and harmoniums).
3. \((++): the transverse (symmetric) model where the Bernoulli pressure causes the valve to close perpendicular to the direction of airflow.

Figure 10: Simplified models of three common configurations of pressure-controlled valves.

Valve Displacement

• Consider the double reed in a blown open configuration.
• Surface \(S_1\) sees an upstream pressure \(p_1\), surface \(S_2\) sees the downstream pressure \(p_2\) (after flow separation), and surface \(S_3\) sees the flow at the interior of the valve channel and the resulting Bernoulli pressure.

Figure 11: Geometry of a blown open pressure-controlled valve showing effective areas \(S_1, S_2, S_3\).
Valve Driving Force

• With these areas and the corresponding geometric couplet defined, the motion of the valve opening $x(t)$ is governed by

$$\frac{d^2x}{dt^2} + 2m\gamma \frac{dx}{dt} + k(x-x_0) = \sigma_1 p_1 (S_1+S_3) + \sigma_2 p_2 S_2,$$

where $\gamma$ is the damping coefficient, $x_0$ the equilibrium position of the valve opening in the absence of flow, $K$ the valve stiffness, and $m$ the reed mass.

• The couplet therefore, is very useful when evaluating the force driving a mode of the vibrating valve.

Discretizing Valve Displacement

• The Laplace Transform of the valve displacement:

$$ms^2X(s) + mgsX(s) + KX(s) - Kx_0 = F(s).$$

• The bilinear transform, defined by the substitution

$$s \rightarrow c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right)$$

where $c = \frac{2}{T}$, yields

$$X(z) + F(z) = \frac{1 + 2z^{-1} + z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}},$$

where

$$a_0 = mc^2 + mgc + k,$$

$$a_1 = -2(mc^2 - k),$$

$$a_2 = mc^2 - mgc + k.$$

• The corresponding difference equation is

$$x(n) = \frac{1}{a_0} [F_k(n) + 2F_k(n-1) + F_k(n-2)] - a_1 x(n-1) - a_2 x(n-2)],$$

where $F_k(n) = F(n) + kx_0$.

Volume Flow

• The steady flow through a valve is determined based on input pressure $p_1$ and the resulting output pressure $p_2$.

• The difference between these two pressures is denoted $\Delta p$ and is related to volume flow via the stationary Bernoulli equation

$$U = A \sqrt{\frac{2\Delta p}{\rho}},$$

where $A$ is the cross section area of the air column (and dependent on the opening $x$).