Music 206: Mechanical Vibration

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Primary Mechanical Vibrators

- With a model of a wind instrument body, a model of the mechanical “primary” resonator is needed for a complete instrument.
- Many sounds are produced by coupling the mechanical vibrations of a source to the resonance of an acoustic tube:
  - In vocal systems, air pressure from the lungs controls the oscillation of a membrane (vocal fold), creating a variable constriction through which air flows.
  - Similarly, blowing into the mouthpiece of a clarinet will cause the reed to vibrate, narrowing and widening the airflow aperture to the bore.

Mass-spring System

- Before modeling this mechanical vibrating system, let’s review some acoustics, and mechanical vibration.
- The simplest oscillator is the mass-spring system:
  - a horizontal spring fixed at one end with a mass connected to the other.

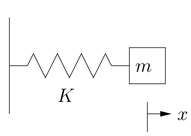


Figure 1: An ideal mass-spring system.

- The motion of an object can be describe in terms of its
  1. displacement $x(t)$
  2. velocity $v(t) = \frac{dx}{dt}$
  3. acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Equation of Motion

- The force on an object having mass m and acceleration a may be determined using *Newton’s second law of motion*

$$F = ma.$$

- There is an elastic force restoring the mass to its equilibrium position, given by *Hooke’s law*

$$F = -Kx,$$

where K is a constant describing the stiffness of the spring.

- The spring force is equal and opposite to the force due to acceleration, yielding the equation of motion:

$$m\frac{d^2x}{dt^2} = -Kx.$$

Solution to Equation of Motion

- The solution to
$$m\frac{d^2x}{dt^2} = -Kx,$$
is a function proportional to its second derivative.
- This condition is met by the sinusoid:
$$\begin{aligned} x &= A\cos(\omega_0 t + \phi) \\ \frac{dx}{dt} &= -\omega_0 A\sin(\omega_0 t + \phi) \\ \frac{d^2x}{dt^2} &= -\omega_0^2 A\cos(\omega_0 t + \phi) \end{aligned}$$
- Substituting into the equation of motion yields:
$$\begin{aligned} m\frac{d^2x}{dt^2} &= -Kx, \\ -\omega_0^2 \cancel{A\cos(\omega_0 t + \phi)} &= -\frac{K}{m}\cancel{A\cos(\omega_0 t + \phi)}, \end{aligned}$$
showing that the natural frequency of vibration is
$$\omega_0 = \sqrt{\frac{K}{m}}.$$

Motion of the Mass Spring

- We can make the following inferences from the oscillatory motion (displacement) of the mass-spring system written as:
$$x(t) = A\cos(\omega_0 t)$$
  - Energy is conserved, the oscillation never decays.
  - *At the peaks*:
    - the spring is maximally compressed or stretched;
    - the mass is momentarily stopped as it is changing direction;
  - *At zero crossings*,
    - the spring is momentarily relaxed (it is neither compressed nor stressed), and thus holds no PE
    - all energy is in the form of KE.
  - Since energy is conserved, the KE at zero crossings is exactly the amount needed to stretch the spring to displacement $-A$ or compress it to $+A$.

PE and KE in the mass-spring oscillator

- All vibrating systems consist of this interplay between an energy storing component and an energy carrying ("massy") component.
- **The potential energy PE** of the ideal mass-spring system is equal to the work done[1] stretching or compressing the spring:
$$\begin{aligned} PE &= \frac{1}{2}Kx^2, \\ &= \frac{1}{2}KA^2\cos^2(\omega_0 t + \phi). \end{aligned}$$
- **The kinetic energy KE** in the system is given by the motion of mass:
$$\begin{aligned} KE &= \frac{1}{2}mv^2, \\ &= \frac{1}{2}\cancel{m\omega_0^2}^{K} A^2\sin^2(\omega_0 t + \phi) \\ &= \frac{1}{2}KA^2\sin^2(\omega_0 t + \phi). \end{aligned}$$

[1] work is the product of the average force and the distance moved in the direction of the force

Conservation of Energy

- The total energy of the ideal mass-spring system is constant:
$$\begin{aligned} E &= PE + KE \\ &= \frac{1}{2}KA^2\left(\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)\right) \\ &= \frac{1}{2}KA^2 \end{aligned}$$

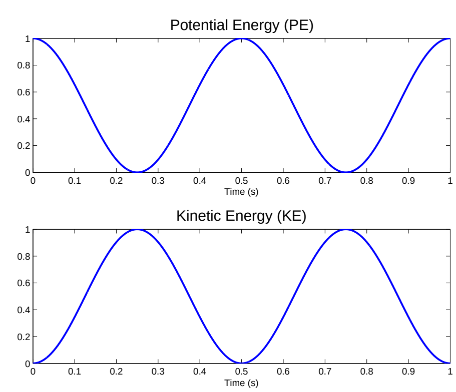


Figure 2: The interplay between PE and KE.

Other Simple Oscillators—Pendulum

- Besides the mass-spring, there are other examples of simple harmonic motion.
- **A pendulum**: a mass m attached to a string of length l vibrates in simple harmonic motion, provided that $x \ll l$.

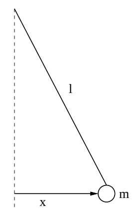


Figure 3: A simple pendulum.

- The frequency of vibration is

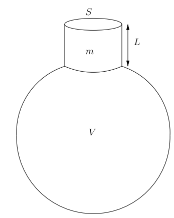
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}},$$

where g is the acceleration due to gravity.

- According to this equation, would changing the mass change the frequency?

Other Simple Oscillators—Helmholtz Resonator

- **A Helmholtz Resonator**: The mass of air in the neck serves as a piston, and the larger volume as the spring.



- The resonant frequency of the Helmholtz resonator is given by

$$f_0 = \frac{c}{2\pi}\sqrt{\frac{a}{VL}},$$

where a is the area of the neck, V is the volume, L is the length of the neck and c is the speed of sound.

Damped Vibration

- In a real system, mechanical energy is lost due to friction and other mechanisms causing *damping*.
- Unless energy is reintroduced into the system, the amplitude of the vibrations with decrease with time.
- A change in peak amplitude with time is called an *envelope*, and if the envelope decreases, the vibrating system is said to be *damped*.

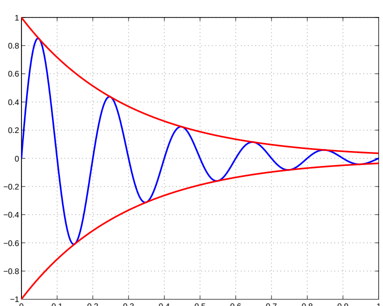


Figure 4: A damped sinusoid.

Mass-spring-damper system

- Damping of an oscillating system corresponds to a loss of energy or equivalently, a decrease in the amplitude of vibration.

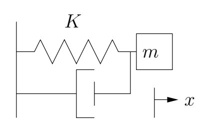


Figure 5: A mass-spring-damper system.

- The damper is a mechanical resistance (or viscosity) and introduces a drag force F_r typically proportional to velocity,

$$\begin{aligned} F_r &= -Rv \\ &= -R\frac{dx}{dt}, \end{aligned}$$

where R is the mechanical resistance.

Damped Equation of Motion

- The equation of motion for the damped system is obtained by adding the drag force into the equation of motion:
$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Kx = 0,$$
or alternatively
$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = 0,$$
where $\alpha = R/2m$ and $\omega_0^2 = K/m$.
- The damping in a system is often measured by the quantity τ, which is the time for the amplitude to decrease to $1/e$:
$$\tau = \frac{1}{\alpha} = \frac{2m}{R}.$$

The Solution to the Damped Vibrator

- The solution to the system equation
$$\frac{dx^2}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = 0$$
has the form
$$x = A(t)\cos(\omega_d t + \phi).$$
where $A(t)$ is the amplitude envelope
$$A(t) = e^{-t/\tau} = e^{-\alpha t},$$
and the natural frequency ω_d
$$\omega_d = \sqrt{\omega_0^2 - \alpha^2},$$
is *lower* than that of the ideal mass-spring system
$$\omega_0 = \sqrt{\frac{K}{m}}.$$
- Peak A and ϕ are determined by the initial displacement and velocity.

Systems with Several Masses

- When there is a single mass, its motion has only one degree of freedom and one natural mode of vibration.
- Consider the system having 2 masses and 3 springs:

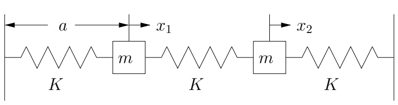


Figure 6: 2-mass 3-spring system.

- The system will have two "normal" independent modes of vibration:
  1. one in which masses move in the *same direction*, with frequency
  $$f_1 = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$
  2. one in which masses move in *different directions* with frequency
  $$f_2 = \frac{1}{2\pi}\sqrt{\frac{3K}{m}}$$

  (assuming equal masses and springs).

System Equations for Two Spring-Coupled Masses

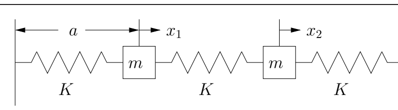


Figure 7: A short section of a string.

- The extensions of the left, middle and right springs are $x_1, x_2 - x_1$, and $-x_2$, respectively.
- When a spring is extended by x, the mass attached to the
  - **left** experiences a *positive* horizontal restoring force $F_r = Kx$;
  - **right** experiences an equal and opposite force $F_r = -Kx$,

  where K is the spring constant.
- The equation of motion for the displacement of the first mass:
$$m\ddot{x}_1 = -Kx_1 + K(x_2 - x_1),$$
and the second mass,
$$m\ddot{x}_2 = -K(x_2 - x_1) + K(-x_2).$$

Additional Modes

- When modes are independent, the system can vibrate in one mode *with minimal* excitation of another.
- Unless constrained to one-dimension, the masses can also move *transversely* (at right-angles to the springs).
- An additional mass adds an additional mode of vibration.
- An N-mass system has N modes per degree of freedom.
- As N gets very large, it becomes convenient to view the system as a continuous string with a uniform mass density and tension.

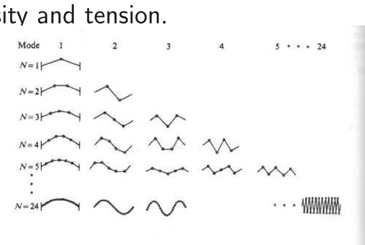


Figure 8: Increasing the number of masses (Science of Sound).

Forced (Driven) Vibration

- Getting they system to vibrate at any single mode of vibration requires exciting, or *driving* the system at the desired mode.
- See Dan Russell's site: The forced harmonic oscillator
- When a simple harmonic oscillator is driven by an external force $F(t)$, the equation of motion becomes

$$m\ddot{x} + R\dot{x} + Kx = F(t).$$

- The driving force may have harmonic time dependence, it may be impulsive, or it can be a random function of time (noise).

Phase of Driven Vibration

- The force driving an oscillation can be illustrated by holding a slinky in the vertical direction.
- Move the hand up and down slowly:
  - At low f_h, *both the hand and the mass move in the same direction, and the spring hardly stretches at all.*
- Increasing f_h makes it harder to move mass, and it lags behind the driving force.
- At resonance $f_h = f_0$:
  - the mass is 1/4 cycle behind the hand.
  - the amplitude is at its maximum.
- The higher the Q (quality factor), of the vibrating system, the more abrupt the transition in phase.

Resonance

- At *resonance*, there is maximum transfer of energy between the hand and the mass-spring system.
- Plotting amplitude A with respect to f_h, shows a curve that is almost symmetrical about its peak A_{max}, i.e., it has a *bandwidth* measured a distance down from A_{max}.

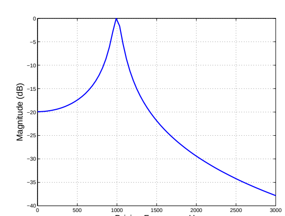


Figure 9: A Resonator shows peak amplitude when the driving force is equal to the system's natural frequency.

- Both the peak A_{max} the bandwidth Δf depend on the damping in the system:
  1. **heavily damped**: Δf is large and A_{max} is small.
  2. **little damping**: Δf is small and A_{max} is large, creating a *sharp* resonance.

Quality Factor and Bandwidth

- The *bandwidth* of the resonance is typically described using the quantity $Q = \frac{f_0}{\Delta f}$, for *quality factor*.
- A high-Q has a sharp resonance, and a low-Q has a broad resonance curve.
- For a vibrator set into motion and left to vibrate freely, its decay time is proportional to the Q of its resonance.
- In terms of our mechanical system, given the resistance $\alpha = \frac{R}{2m}$, the quality factor is:

$$Q = \frac{\omega_0}{2\alpha}.$$

How to Discretize?

- The one-sided Laplace transform of a signal $x(t)$ is defined by

$$X(s) \triangleq \mathcal{L}_s\{x\} \triangleq \int_0^\infty x(t)e^{-st}dt$$

where t is real and $s = \sigma + j\omega$ is a complex variable.

- The differentiation theorem for Laplace transforms states that

$$\frac{d}{dt}x(t) \leftrightarrow sX(s)$$

where $x(t)$ is any differentiable function that approaches zero as t goes to infinity.

- The transfer function of an ideal differentiator is $H(s) = s$, which can be viewed as the Laplace transform of the operator d/dt.
- Given the equation of motion

$$m\ddot{x} + R\dot{x} + Kx = F(t),$$

the Laplace Transform is

$$s^2X(s) + 2\alpha sX(s) + \omega_0^2X(s) = F(s).$$

Finite Difference

- The finite difference approximation (FDA) amounts to replacing derivatives by finite differences, or

$$\frac{d}{dt}x(t) \triangleq \lim_{\delta\to 0}\frac{x(t) - x(t-\delta)}{\delta} \approx \frac{x(nT) - x[(n-1)T]}{T}.$$

- The z transform of the first-order difference operator is $(1 - z^{-1})/T$. Thus, in the frequency domain, the finite-difference approximation may be performed by making the substitution

$$s \to \frac{1 - z^{-1}}{T}$$

- The first-order difference is first-order error accurate in T. Better performance can be obtained using the bilinear transform, defined by the substitution

$$s \longrightarrow c\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \qquad \text{where} \quad c = \frac{2}{T}.$$

FDA of Equation of Motion

Pressure Controlled Valves

- Returning now to our model of wind instruments....
- Blowing into an instrument mouthpiece creates a pressure difference across the surface of the reed.
- When the reed oscillates, it creates and alternating opening and closure to the bore, allowing airflow entry during the open phase and cutting it off during the closed phase.
- The effect, is often seen as a periodic train of pressure pulses into the bore.
- Sound sources of this kind are referred to as pressure-controlled valves and they have been simulated in various ways to create musical synthesis models and vocal systems.

Classifying Pressure-Controlled Valves

- The method for simulating the reed typically depends on whether an additional upstream or downstream pressure causes the corresponding side of the valve to open or close further.
- As per Fletcher, the couplet (σ_1, σ_2) may be used describe the upstream and downstream valve behaviour, respectively.
- A σ_n value of
  - $+1$ indicates an **opening** of the valve,
  - -1 indicates a **closing** of the value,

  on side n in response to a pressure increase.

Three simple configurations of PC valves

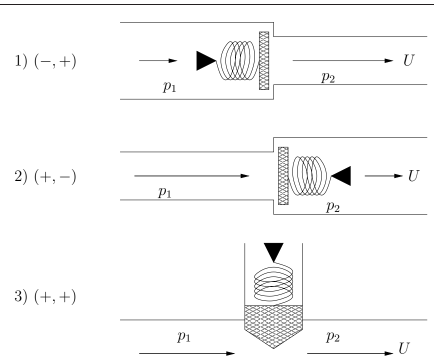


Figure 10: Simplified models of three common configurations of pressure-controlled valves.

1. $(-,+)$: the valve is blown closed (as in woodwind instruments or reed-pipes of the pipe organ).
2. $(+,-)$: the valve is blown open (as in the simple lip-reed models for brass instruments, the human larynx, harmonicas and harmoniums).
3. $(+,+)$: the transverse (symmetric) model where the Bernoulli pressure causes the valve to close perpendicular to the direction of airflow.

Valve Displacement

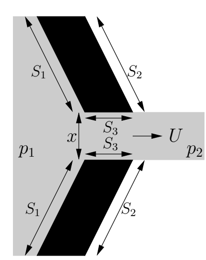


Figure 11: Geometry of a blown open pressure-controlled valve showing effective areas S_1, S_2, S_3.

- Consider the double reed in a blown open configuration.
- Surface S_1 sees an upstream pressure p_1, surface S_2 sees the downstream pressure p_2 (after flow separation), and surface S_3 sees the flow at the interior of the valve channel and the resulting Bernoulli pressure.

Valve Driving Force

- With these areas and the corresponding geometric couplet defined, the motion of the valve opening $x(t)$ is governed by

$$m\frac{d^2x}{dt^2}+2m\gamma\frac{dx}{dt}+k(x-x_0)=\sigma_1 p_1(S_1+S_3)+\sigma_2 p_2 S_2,$$

where γ is the damping coefficient, x_0 the equilibrium position of the valve opening in the absence of flow, K the valve stiffness, and m the reed mass.

- The couplet therefore, is very useful when evaluating the force driving a mode of the vibrating valve.

Discretizing Valve Displacement

- The Laplace Transform of the valve displacement:

$$ms^2X(s)+mgsX(s)+KX(s)-Kx_0=F(s).$$

- The bilinear transform, defined by the substitution

$$s \longrightarrow c\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad \text{where} \quad c=\frac{2}{T},$$

yields

$$\frac{X(z)}{F(z)+kx_0}=\frac{1+2z^{-1}+z^{-2}}{a_0+a_1z^{-1}+a_2z^{-2}},$$

where

$$\begin{aligned} a_0 &= mc^2+mgc+k, \\ a_1 &= -2(mc^2-k), \\ a_2 &= mc^2-mgc+k. \end{aligned}$$

- The corresponding difference equation is

$$\begin{aligned} x(n) \;=\; & \frac{1}{a_0}[F_k(n)+2F_k(n-1)+F_k(n-2)) - \\ & a_1x(n-1)-a_2x(n-2)], \end{aligned}$$

where $F_k(n)=F(n)+kx_0$.

Volume Flow

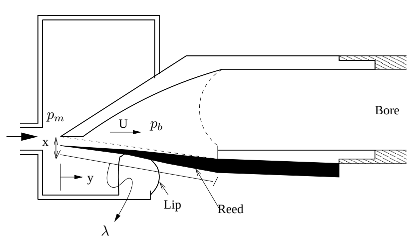


Figure 12: The clarinet Reed.

- The steady flow through a valve is determined based on input pressure p_1 and the resulting output pressure p_2.
- The difference between these two pressure is denoted Δp and is related to volume flow via the stationary Bernoulli equation

$$U=A\sqrt{\frac{2\Delta p}{\rho}},$$

where A is the cross section area of the air column (and dependent on the opening x).

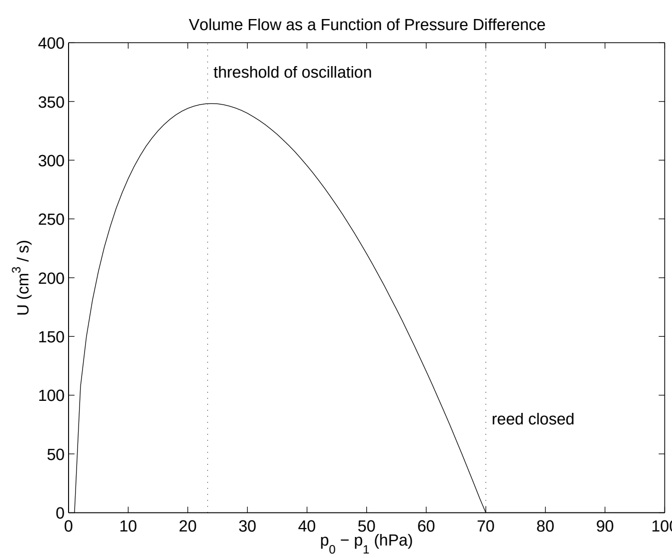


Figure 13: The reed table.