Music 206: Modeling Acoustic Tubes and Wind Instrument Bores/Bells

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A Digital Waveguide Section

- Both **plane waves** of a cylinder and **spherical waves** of a cone can be modeled using a digital waveguide.
- For an infinitely long tube, filter $\lambda(\omega)$ accounts for the one-way propagation loss.

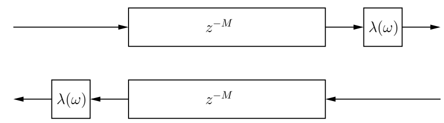


Figure 1: A waveguide section for an acoustic tube showing delay and wall loss $\lambda(\omega)$.

- For a cone, an addtional filter is required to account for spherical spreading at a distance ρ from the cone apex:

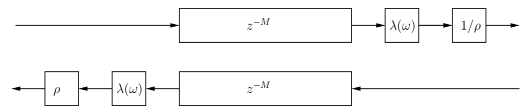


Figure 2: A waveguide section for a cone showing the additional spherical spreading ρ.

Simple Wall Loss

- The effects of viscous drag and thermal conduction along the bore walls lead to an attenuation in the propagating waves given by

$$\alpha(\omega) = 2 \times 10^{-5} \frac{\sqrt{\omega}}{a}$$

where a is the bore radius, valid for tube sizes seen in most musical instruments.

- The round-trip attenuation for a tube of length L is given by

$$\lambda^2(\omega) = e^{-2\alpha L}$$

Termination and Scattering

- A change of impedance, such as a termination or connection to another waveguide section, will require additional filters to model any reflection and transmission.

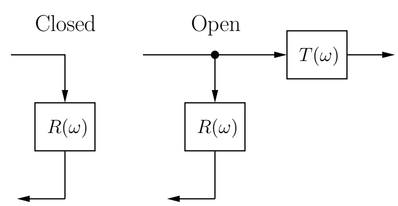


Figure 3: Termination: closed and open ends.

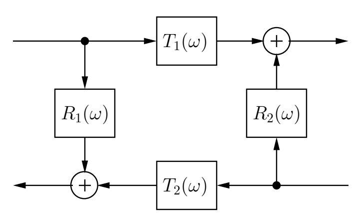


Figure 4: Scattering: a junction with another section.

Change in Tube Area

- A change in tube cross-sectional area creates a junction between two impedances Z_1 and Z_2.

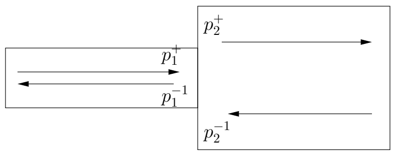


Figure 5: Tube with a change in cross-sectional area.

- The pressure and velocity at the junction is given by

$$\begin{aligned} p(J) &= p_1^+(J) + p_1^-(J) \\ U(J) &= U_1^+(J) + U_1^-(J) \\ &= \frac{1}{Z_1}(p_1^+(J) - p_1^-(J)) \end{aligned}$$

where

$$p_1^+(J) = Z_1 U_1^+, \quad p_1^-(J) = -Z_1 U_1^-.$$

- The new impedance at the junction Z_2, is given by

$$Z_2 = \frac{p(J)}{U(J)} = Z_1 \frac{p_1^+(J) + p_1^-(J)}{p_1^+(J) - p_1^-(J)}.$$

Deriving the Reflection Coefficient

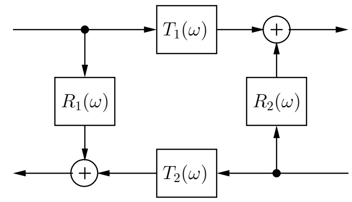


Figure 6: Scattering: a junction with another section.

- The relationship between the adjacent impedances causes a **reflection** in the first section that can be expressed in terms of tube area cross-sectional S,

$$\begin{aligned} R_1 &= \frac{p_1^-(J)}{p_1^+(J)} \\ &= \frac{Z_2 - Z_1}{Z_2 + Z_1} \\ &= \frac{S_1 - S_2}{S_1 + S_2} = k. \end{aligned}$$

- It follows that the reflection in the second section is

$$R_2 = \frac{S_2 - S_1}{S_2 + S_1} - = -k.$$

Two-port Scattering Junction

- The junction reflection filters are given by

$$R_1 = k \qquad \text{and} \qquad R_2 = -k.$$

- For cylinders, what is not reflected is transmitted:
  - the transmission filters are **amplitude complementary**,
  - for pressure waves yields:

$$T_1 = 1 + R_1 = 1 + k,$$

and

$$T_2 = 1 + R_2 = 1 - k.$$

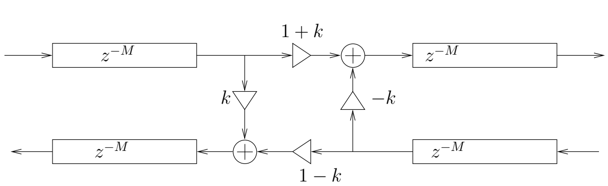


Figure 7: A two port junction between two cylinders.

Impedance of Conical Sections

- If one of the waveguide sections is a cone, then impedance is actually **frequency dependent**.
- For spherical pressure waves in conical tubes, waves **popagating away from the cone apex** (denoted by the $+$ superscript), the impedance is given by

$$Z_n^+(r;\omega) = \frac{\rho c}{S} \frac{j\omega}{j\omega + c/r}, \qquad (1)$$

where r is the distance from the observation point to the cone apex.

- For spherical waves **propagating toward the cone apex**, the impedance is given by

$$Z_n^-(r;\omega) = \frac{\rho c}{S} \frac{j\omega}{j\omega - c/r} = Z_n^{+*}(r;\omega). \qquad (2)$$

Open-End Reflection

- A special condition in a change of impedance occurs at the open end of an acoustic tube.

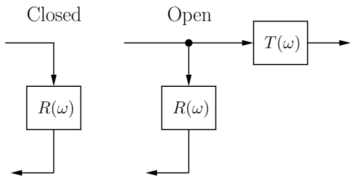


Figure 8: Termination: closed and open ends.

- The reflection filter for an open end is given by

$$R_{op}(\omega) = \frac{Z_L(\omega)/Z_0 - 1}{Z_L(\omega)/Z_0 + 1},$$

where $Z_0 = \frac{\rho c}{S}$ is the wave impedance, and $Z_L(\omega)$ is the complex terminating impedance at the open end.

- The open-end impedance $Z_L(\omega)$ is rather complicated, but an expression is given for cylindrical tubes by Levine and Schwinger in terms of Bessel and Struve functions [1].

[1] see "On the Radiation of Sound from an Unflanged Circular Pipe", *Physical Review*, vol. 73, no.4, February 15 1948

Open-End Reflection Function

- From the data plots given by Levine and Schwinger, it turns out the ratio Z_L/Z_0 may be approximated by

$$Z_L/Z_0 \approx \frac{jka}{\zeta + jka}$$

where $k = \omega/c$ is the wavenumber, and a is the radius of the cylinder, and ζ is a scalar near one[2].

- Substituting this expression for Z_L/Z_1 into

$$R_{op}(\omega) = \frac{Z_L(\omega)/Z_0 - 1}{Z_L(\omega)/Z_0 + 1},$$

yields a reflection filter approximated by

$$R_{op} = \frac{-1}{1 + 2jka/\zeta}.$$

- The open-end reflection filter R_{op} is a one-pole filter with a cut-off frequency of $\omega = \zeta c/(2a)$.

[2] The scalar determines the transition between the low-frequency and high-frequency behaviour.

Open-End Reflection Filter Design

```
N = 1024;
fs = 44100;
c = 340; % sound speed
a = 0.01; % bore radius
omega = 2*pi*[0:N-1]'*fs/2/N;
k = omega/c; zeta = .9;
jka = j.*k.*a;
ZLoZ1 = jka./(zeta + jka);
R = (ZLoZ1 - 1)./(ZLoZ1 + 1);

[Bi,Ai] = invfreqz(R, linspace(0,pi,N), 1,1);
[Bs,As] = stmcb(real(ifft(R)), 1,1);
[Bp,Ap] = prony(real(ifft(R)), 1,1);
```

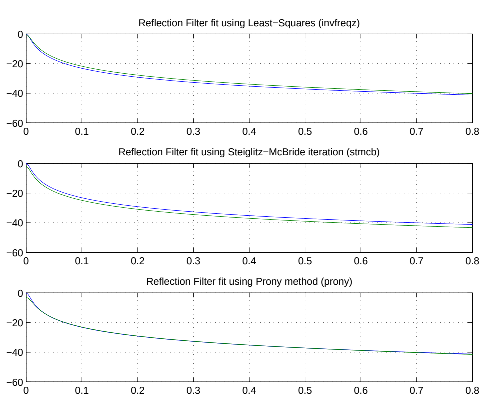


Figure 9: Several filter design methods to fit a filter.

Scattering

- **Scattering** is a phenomenon in which the wave is *scattered* into an infinity of waves propagating in different directions.
- As we have seen, a discontinuity in the diameter of an acoustic tube will cause traveling waves to be partially reflected and partially transmitted—a *scattering* in two (2) directions.
- Examples of discontinuities causing scattering:
  - A change of tube diameter
  - A junction with other acoustic elements (tubes, cavities),
  - Tone holes.

N-port Parallel Scattering Junctions

- All scattering junctions arise from the parallel connection of N physical waveguides.
- Consider the junction of N lossless acoustic tubes, each with real, positive, scalar wave admittance

$$\Gamma_n = \frac{1}{Z_n}.$$

where Z_n is the wave impedance of the n^{th} waveguide.
- Each port on the junction has both an input and output terminal.
- The physical pressure and velocity at a port n is equal to the sum of the corresponding incoming and outgoing variables, that is,

$$\begin{aligned} p_n &= p_n^+ + p_n^- \\ U_n &= U_n^+ + U_n^-. \end{aligned}$$

Physical Variables at the Junction

- The **law for conservation of mass and momentum** dictate that *the pressure at the junction must be continuous*.
- Therefore, for a junction with N ports, the junction pressure p_J is given by

$$p_J = p_n, \qquad n = 1 \ldots N.$$

- By the same law, *the volume flow at the N-port junction sums to zero*,

$$\sum_{n=1}^{N} U_n = 0.$$

The pressure at the junction is the same for all N waveguides at the junction point, and the velocities from each branch sum to zero.

Junction Pressure

- Because of pressure continuity at the junction, the outgoing pressure on any port n is given by

$$\begin{aligned} p_J &= p_n \\ &= p_n^+ + p_n^- \\ p_n^- &= p_J - p_n^+. \end{aligned}$$

- Using the fact that velocities from each port sum to zero,

$$\sum_{n=1}^{N} \left(U_n^+ + U_n^-\right) = 0,$$

substituting the relationship to pressure traveling wave components,

$$\sum_{n=1}^{N} \left(\frac{p_n^+}{Z_n} - \frac{p_n^-}{Z_n}\right) = 0,$$

as well as the outgoing pressure on any port n,

$$\sum_{n=1}^{N} \left(\frac{p_n^+}{Z_n} - \frac{p_J - p_n^+}{Z_n}\right) = 0,$$

we may solve for the pressure at the junction

$$\begin{aligned} \sum_{n=1}^{N} \frac{2p_n^+}{Z_n} &= \sum_{n=1}^{N} \frac{p_J}{Z_n} \\ 2\sum_{n=1}^{N} \frac{p_n^+}{Z_n} &= p_J \sum_{n=1}^{N} \frac{1}{Z_n} \\ p_J &= \frac{2\sum \Gamma_n p_n^+}{\sum \Gamma_n} \end{aligned}$$

where $\Gamma_n = 1/Z_n$ is the characteristic admittance of the n^{th} port.

Matrix Formulation of the N-port

- If

$$\mathbf{p}^+ = \begin{bmatrix} p_1^+ \\ p_2^+ \\ \cdot \\ \cdot \\ p_N^+ \end{bmatrix}, \quad \mathbf{p}^- = \begin{bmatrix} p_1^- \\ p_2^- \\ \cdot \\ \cdot \\ p_N^- \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \frac{1}{Z_1^+} & \frac{1}{Z_2^+} & \frac{1}{Z_3^+} & \frac{1}{Z_4^+} & \dots & \frac{1}{Z_{N-1}^+} & \frac{1}{Z_N^+} \end{bmatrix},$$

$$\mathbf{B} = -\begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \frac{1}{Z_1^-} & \frac{1}{Z_2^-} & \frac{1}{Z_3^-} & \frac{1}{Z_4^-} & \dots & \frac{1}{Z_{N-1}^-} & \frac{1}{Z_N^-} \end{bmatrix},$$

then we can write

$$\mathbf{A}\mathbf{p}^+ = \mathbf{B}\mathbf{p}^-,$$

and

$$\mathbf{p}^- = \mathbf{B}^{-1}\mathbf{A}\mathbf{p}^+.$$

Three-Port Parallel Junction

- In an example of three connecting tubes, the model would require three digital waveguides and a 3-port parallel junction to simulate the point at which they connect.

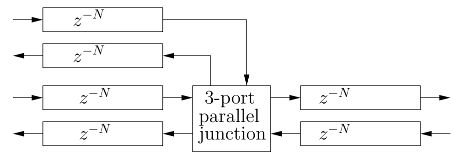


Figure 10: Three digital waveguides meeting at a three-port parallel junction.

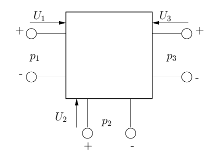


Figure 11: Three-Port Parallel Junction.

- The model of the three-port parallel junction will receive an incoming pressure wave, p_n^+, on each of the three ports, and will return the corresponding outgoing pressure value,

$$p_n^- = p_J - p_n^+.$$

Example uses of the 3-port junction

- Tone holes in acoustic wind instruments: three connected tubes, with the finger hole being a very short branch off the main bore (Douglas Keefe).
- The bifurcation at the velum in the human vocal tract: simulating the oral and nasal airways (P. Cook).
- The bifurcations of the trachea to the left and right bronchi: simulating the avian syrinx (Tamara Smyth).

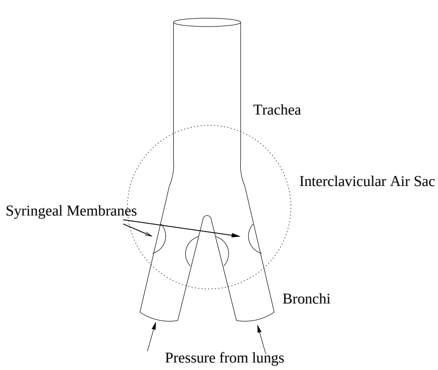


Figure 12: Syrinx

Modeling Change in Cross-Section

- The human vocal tract has different profiles when producing different vowel sounds.

Figure 13: 'ae' as in "bat".

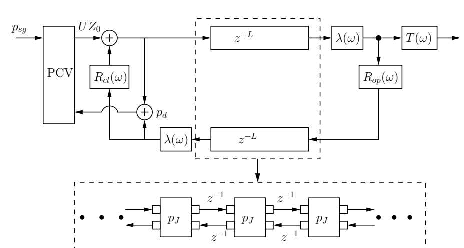


Figure 14: Modeling a change in cross section using a sequence of 2-port scattering junctions.

- A sequence of two-port scattering junctions can model a piecewise approximation of a tube changing in cross section.

Modeling the Instrument Bell

- Another example, is the continually changing cross-section of an instrument bell.

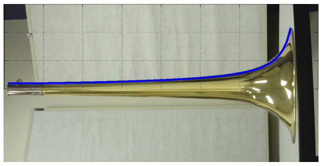


Figure 15: The profile of a trombone bell.

- The tombone bell is well described by the so-called Bessel horn,
$$a(x) = b(x + x_0)^{-\gamma}, \qquad (3)$$
where x_0 is the position of the mouth of the horn, x is the distance from the horn mouth, and $a(x)$ is the radius over the length of the bell.
- For more efficient computation, the effects of the bell may be *lumped* to a single reflection and transmission function.

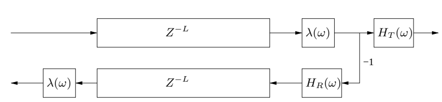
Obtaining Bell Reflection Function

- The relationship between traveling waves in adjacent sections may be written in matrix form as
$$\begin{bmatrix} p_n^+ \\ p_n^- \end{bmatrix} = \mathbf{A_n} \begin{bmatrix} p_{n+1}^+ \\ p_{n+1}^- \end{bmatrix},$$
where the scattering matrix is given by
$$\mathbf{A_n} = \begin{bmatrix} \frac{Z_n}{Z_{n+1}} \frac{Z_{n+1}+Z_n^*}{Z_n+Z_n^*} e^{jkL_{n+1}} & \frac{Z_n}{Z_{n+1}^*} \frac{Z_{n+1}^*-Z_n^*}{Z_n+Z_n^*} e^{-jkL_{n+1}} \\ \frac{Z_n^*}{Z_{n+1}} \frac{Z_{n+1}-Z_n}{Z_n+Z_n^*} e^{jkL_{n+1}} & \frac{Z_n^*}{Z_{n+1}^*} \frac{Z_{n+1}^*+Z_n}{Z_n+Z_n^*} e^{-jkL_{n+1}} \end{bmatrix},$$
for a section of length L_n and with complex wave impedance Z_n.
- For a model having N sections, $N-1$ scattering matrices are multiplied,
$$\begin{bmatrix} p_1^+ \\ p_1^- \end{bmatrix} = \mathbf{A_1 A_2 \ldots A_{N-1}} \begin{bmatrix} p_N^+ \\ p_N^- \end{bmatrix},$$
to yield the model's single final scattering matrix
$$\mathbf{P} = \prod_{n=1}^{N-1} \mathbf{A_n},$$
relating the bell input and output traveling pressure waves.

- The expression for the reflection function of the bell may be formed by taking the ratio of the wave reflected by the bell p_1^- to the bell input wave p_1^+,
$$\begin{aligned} R_B = \frac{p_1^-}{p_1^+} &= \lambda^2(\omega) \frac{p_N^+ \mathbf{P_{2,1}} + p_N^- \mathbf{P_{2,2}}}{p_N^+ \mathbf{P_{1,1}} + p_N^- \mathbf{P_{1,2}}} \\ &= \lambda^2(\omega) \frac{\mathbf{P_{2,1}} + \mathbf{P_{2,2}} R_L(\omega)}{\mathbf{P_{1,1}} + \mathbf{P_{1,2}} R_L(\omega)}, \end{aligned}$$
where the final expression is obtained by incorporating an open end reflection at the termination of the N^{th} section by substituting $p_N^- = p_N^+ R_L(\omega)$, and by commuting round-trip propagation losses $\lambda^2(\omega)$.
- Similarly, the bell transmission is given by the ratio of the wave radiated out the bell $p_N^+ T_L(\omega)$, where $T_L \omega)$ is the open-end transmission function, to the bell's input p_1^+,
$$T_B(\omega) = \frac{p_N^+ \lambda(\omega) T_L(\omega)}{p_1^+} = \frac{\lambda(\omega) T_L(\omega)}{\mathbf{P_{1,1}} + \mathbf{P_{1,2}} R_L(\omega)}. \qquad (4)$$

Wall Losses

- In practical implementations of acoustic tubes using digital waveguide synthesis, it is necessary to account for the losses associated with **viscous drag** and **thermal conduction** which take place primarily within a thin boundary layer along the bore walls.
- Below is a waveguide model of a cylindrical tube with commuted wall loss filters, $\lambda(\omega)$, a reflection filter $H_R(\omega)$ and a transmission filter $H_T(\lambda)$.



- Though these losses are *distributed* over the length of the tube, it is more efficient to lump these effects by *commuting* a characteristic digital filter to each end of the waveguide.

Simple Wall Loss

- It is possible to account for attenuation due to wall losses simply by multiplying delay line outputs by a single coefficient β.
- For a tube of length L and radius a, with observation points at the ends of the upper and lower delay lines, the constant β has the following approximate value:

$$\beta \approx 1 - 2\alpha L, \qquad \alpha \approx 2 \times 10^{-5}\omega^{\frac{1}{2}}/a,$$

  where ω is the radian frequency. This approximation is valid for tubes sufficiently short that β is near one.
- Since losses are frequency dependent however, this approximation may not be satisfactory for all tubes (large and small).

Wall Loss and Pipe Radius

- The walls contribute a **viscous drag** dependent on the ratio of the pipe radius a, given by the parameter r_v.
- Losses due to **thermal conduction** are also dependent on the pipe radius and are given by the parameter r_t.
- The effects of the viscous and thermal losses lead to an attenuation in the waves propagating down the length of the pipe. The propagation constant is given by

$$\Gamma(\omega) = \alpha(\omega) + j\frac{\omega}{v(\omega)},$$

  where ω is the radian frequency and

$$\alpha(\omega) \triangleq \text{the attenuation coefficient,}$$
$$v(\omega) \triangleq \text{the phase velocity}$$

- The attenuation coefficient $\alpha(\omega)$ is a function of r_v and the phase velocity $v(\omega)$ is a function of r_t. Both therefore, are dependent on the tube radius.[3]

[3] See "A Simple, Accurate wall loss filter for acoustic tubes", by Jonthan Abel and Tamara Smyth, DAFX 2003, for details.

Frequency Response

- The propagation constant $\Gamma(\omega)$ gives the per unit length attenuation of waves propagating along an infinitely long tube.
- The frequency response approximating the attenuation and phase delay over a tube of length L is given by

$$e^{-\Gamma(\omega)L} = e^{-\alpha(\omega)L - j(\omega/v(\omega))L}.$$

- Removing a pure delay of duration L/c, we have the desired wall loss filter frequency response $\lambda(\omega)$, given by

$$\lambda(\omega) = e^{-\alpha(\omega)L - j(\omega/v(\omega) - \omega/c)L}.$$

- The wall loss filter $\lambda(\omega)$ is seen to have a gentle low-pass characteristic, which is more pronounced with decreasing tube radius a or increasing tube length L.

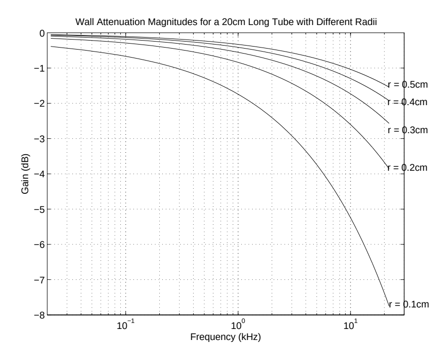


Figure 16: Wall loss filter magnitude for increasing radii.

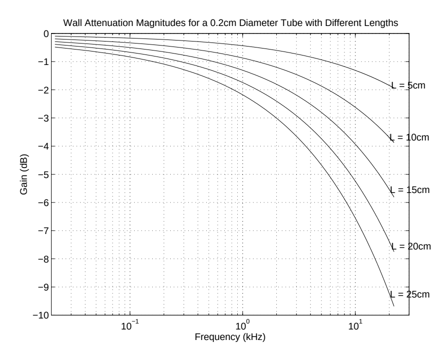


Figure 17: Wall loss filter magnitude for increasing lengths.

Shelf Filter

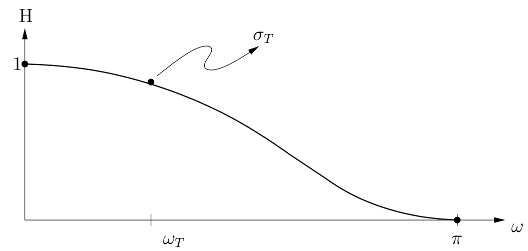


Figure 18: First-order shelf filter where $H(\omega) = 0$ at $\omega = \pi$ and $H(\omega) = 1$ at $\omega = 0$. The cutoff frequency is given by ω_T and $H(\omega_T)$ is given by σ_T.

- The first-order shelf is characterized by
  1. Unity gain at DC
  2. A band edge gain $g_\pi > 0$
  3. A gain $\sqrt{g_\pi}$ at the transition frequency f_t.

First Order Shelf Filter Design

- The transfer function for the first-order shelf filter is given by

$$\sigma(z; f_t, g_\pi) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}.$$

- The coefficients are given by

$$b_0 = \frac{\beta_0 + \rho\beta_1}{1 + \rho\alpha_1}, \quad b_1 = \frac{\beta_1 + \rho\beta_0}{1 + \rho\alpha_1}, \quad a_1 = \frac{\rho + \alpha_1}{1 + \rho\alpha_1},$$

where

$$\beta_0 = \frac{(1 + g_\pi) + (1 - g_\pi)\alpha_1}{2}$$
$$\beta_1 = \frac{(1 - g_\pi) + (1 + g_\pi)\alpha_1}{2},$$

and

$$\alpha_1 = \begin{cases} 0, & g_\pi = 1 \\ \eta - \text{sign}(\eta)(\eta^2 - 1)^{\frac{1}{2}}, & g_\pi \neq 1. \end{cases}$$

- To form the coefficients of a prototype shelf filter with a transition frequency of $\pi/2$

$$\eta = \frac{(g_\pi + 1)}{(g_\pi - 1)}, \quad g_\pi \neq 1.$$

- The parameter

$$\rho = \frac{\sin(\pi f_t/2 - \pi/4)}{\sin(\pi f_t/2 + \pi/4)}$$

is the coefficient of the first-order allpass transformation warping the prototype filter to the proper transition frequency.

- An efficient way to model the frequency response is to use a cascade of minimum-phase first-order shelf filters $\sigma_i(z)$,

$$\hat{\lambda}(z) = \prod_{i=1}^{N} \sigma_i(z; f_t(i), g_\pi(i)).$$

- Unlike with other methods, the shelf filter coefficients are easily computed in real time.
- Since the shelf filters are first order, they have real poles and zeros, and are relatively free of artifacts when made time varying.
- In designing the prototype filters for different orders it was discovered that a cascade of shelf filters with band-edge gains (in units of amplitude) and transition

frequencies (in radians/π) given by

$$g_\pi(i) = \exp\left\{\frac{[(i - \frac{1}{2})/N]^{\frac{1}{2}}}{\sum_{k=1}^{N}[(k - \frac{1}{2})/N]^{\frac{1}{2}}} \cdot \ln|\lambda(2\pi f_s/2)|\right\},$$
$$f_t(i) = [(i - \frac{1}{2})/N]^3,$$

has a transfer function which is an excellent approximation to $\lambda(\omega)$ for a wide range of filter orders N, tube lengths L, tube radii a, and sampling rates f_s.

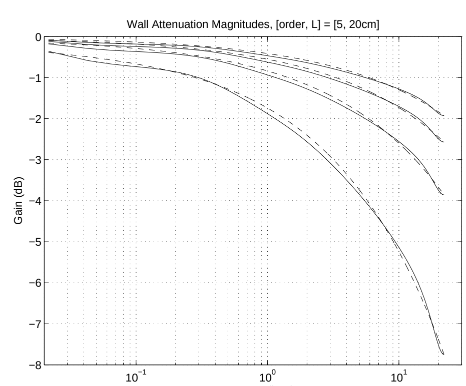


Figure 19: Computed and shelf filter cascade wall loss filter magnitude at various tube radii: 0.1 (lower) - 0.5 (upper) cm.

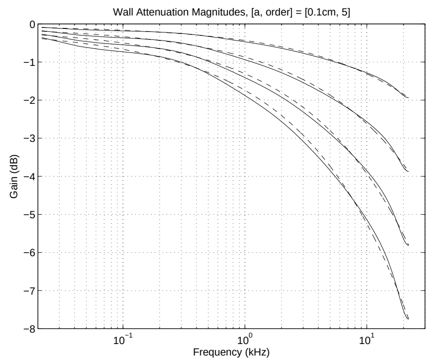


Figure 20: Computed and shelf filter cascade wall loss filter magnitude at various tube lengths: 5 (upper) - 25 (lower) cm.

Pressure Controlled Valves

- Many sounds are produced by coupling the mechanical vibrations of a source to the resonance of an acoustic tube:
- In vocal systems, air pressure from the lungs controls the oscillation of a membrane (or vocal fold), creating a variable constriction through which air flows.
- Similarly, blowing into the mouthpiece of a clarinet will cause the reed to vibrate, narrowing and widening the airflow aperture to the bore.
- Sound sources of this kind are referred to as pressure-controlled valves and they have been simulated in various ways to create musical synthesis models of woodwind and brass instruments as well as animal vocal systems.

Classifying Pressure-Controlled Valves

- In physical modeling synthesis, the method used for simulating the reed typically depends on whether an additional upstream or downstream pressure causes the corresponding side of the valve to open or close further.
- The couplet (σ_1, σ_2) may be used describe the upstream and downstream valve behaviour, respectively, where a value of $+1$ indicates an opening of the valve, and a value of -1 indicates a closing of the value, in response to a pressure increase.

Three simple configurations of PC valves

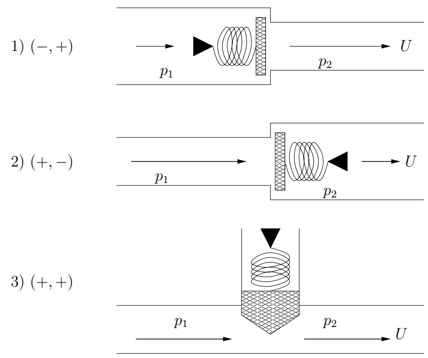


Figure 21: Simplified models of three common configurations of pressure-controlled valves.

1. $(-,+)$: the valve is blown closed (as in woodwind instruments or reed-pipes of the pipe organ).
2. $(+,-)$: the valve is blown open (as in the simple lip-reed models for brass instruments, the human larynx, harmonicas and harmoniums).
3. $(+,+)$: the transverse (symmetric) model where the Bernoulli pressure causes the valve to close perpendicular to the direction of airflow.

Mechanical Vibrations: Mass-spring System

- The simplest oscillator is the mass-spring system which consists of a horizontal spring fixed at one end with a mass connected at the other end.

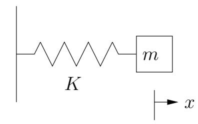


Figure 22: An ideal mass-spring system.

- The motion of an object can be describe in terms of its
  1. displacement $x(t)$
  2. velocity $v(t) = \frac{dx}{dt}$
  3. acceleration $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Equation of Motion

- The force on an object may be determined using *Newton's second law of motion* $F = ma$.
- There is an elastic force restoring the mass to its equilibrium position, given by Hooke's law $F = -Kx$, where K is a constant describing the stiffness of the spring.
- Since Newton's third law of motion states that "for every action there is an equal and opposite reaction', these forces are equal, yielding

$$m\frac{d^2x}{dt^2} = -Kx \quad \longrightarrow \quad \frac{d^2x}{dt^2} + \omega_0^2 x = 0,$$

  where $\omega_0 = \sqrt{K/m}$.
- Recall that $x = A\cos(\omega_0 t + \phi)$ is a solution to this equation.

Damping

- Any real vibrating system tends to lose mechanical energy as a result of friction and other loss mechanisms.
- Damping of an oscillating system corresponds to a loss of energy or equivalently, a decrease in the amplitude of vibration.

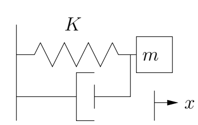


Figure 23: A mass-spring-damper system.

- The drag force F_r is proportional to the velocity and is given by

$$F_r = -Rv \qquad \longrightarrow \qquad F_r = -R\frac{dx}{dt}$$

  where R is the mechanical resistance.

Damped Equation of Motion

- The equation of motion for the damped system is obtained by adding the drag force into the equation of motion:
$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Kx = 0,$$
or alternatively
$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = 0,$$
where $\alpha = R/2m$ and $\omega_0^2 = K/m$.
- The damping in a system is often measured by the quantity τ, which is the time for the amplitude to decrease to $1/e$:
$$\tau = \frac{1}{\alpha} = \frac{2m}{R}.$$
- When a simple oscillator is driven by an external force $F(t)$, the equation of motion becomes
$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Kx = F(t).$$

Valve Displacement

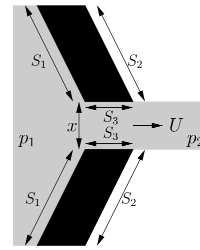


Figure 24: Geometry of a blown open pressure-controlled valve showing effective areas S_1, S_2, S_3.

- Consider the double reed in a blown open configuration.
- Surface S_1 sees an upstream pressure p_1, surface S_2 sees the downstream pressure p_2 (after flow separation), and surface S_3 sees the flow at the interior of the valve channel and the resulting Bernoulli pressure.

Valve Driving Force

- With these areas and the corresponding geometric couplet defined, the motion of the valve opening $x(t)$ is governed by
$$m\frac{d^2x}{dt^2} + 2m\gamma\frac{dx}{dt} + k(x - x_0) = \sigma_1 p_1(S_1 + S_3) + \sigma_2 p_2 S_2,$$
where γ is the damping coefficient, x_0 the equilibrium position of the valve opening in the absence of flow, K the valve stiffness, and m the reed mass.
- The couplet therefore, is very useful when evaluating the force driving a mode of the vibrating valve.

How to Discretize?

- The one-sided Laplace transform of a signal $x(t)$ is defined by
$$X(s) \triangleq \mathcal{L}_s\{x\} \triangleq \int_0^\infty x(t)e^{-st}dt$$
where t is real and $s = \sigma + j\omega$ is a complex variable.
- The differentiation theorem for Laplace transforms states that
$$\frac{d}{dt}x(t) \leftrightarrow sX(s)$$
where $x(t)$ is any differentiable function that approaches zero as t goes to infinity.
- The transfer function of an ideal differentiator is $H(s) = s$, which can be viewed as the Laplace transform of the operator d/dt.
- Taking the Laplace Transform of the valve displacement, we have
$$ms^2X(s) + mgsX(s) + KX(s) - Kx_0 = F(s).$$

Finite Difference

- The finite difference approximation (FDA) amounts to replacing derivatives by finite differences, or

$$\frac{d}{dt}x(t) \triangleq \lim_{\delta \to 0} \frac{x(t) - x(t-\delta)}{\delta} \approx \frac{x(nT) - x[(n-1)T]}{T}.$$

- The z transform of the first-order difference operator is $(1 - z^{-1})/T$. Thus, in the frequency domain, the finite-difference approximation may be performed by making the substitution

$$s \to \frac{1 - z^{-1}}{T}$$

- The first-order difference is first-order error accurate in T. Better performance can be obtained using the bilinear transform, defined by the substitution

$$s \longrightarrow c\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \qquad \text{where} \quad c = \frac{2}{T}.$$

Valve Transfer Function

- Using the Bilinear Transform, the transfer function becomes

$$\frac{X(z)}{F(z) + kx_0} = \frac{1 + 2z^{-1} + z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}},$$

where

$$\begin{aligned} a_0 &= mc^2 + mgc + k, \\ a_1 &= -2(mc^2 - k), \\ a_2 &= mc^2 - mgc + k. \end{aligned}$$

- The corresponding difference equation is

$$\begin{aligned} x(n) &= \frac{1}{a_0}[F_k(n) + 2F_k(n-1) + F_k(n-2)) - \\ &\quad a_1 x(n-1) - a_2 x(n-2)], \end{aligned}$$

where $F_k(n) = F(n) + kx_0$.

Volume Flow

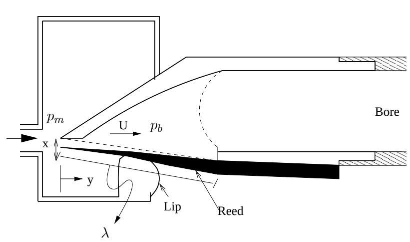


Figure 25: The clarinet Reed.

- The steady flow through a valve is determined based on input pressure p_1 and the resulting output pressure p_2.
- The difference between these two pressure is denoted Δp and is related to volume flow via the stationary Bernoulli equation

$$U = A\sqrt{\frac{2\Delta p}{\rho}},$$

where A is the cross section area of the air column (and dependent on the opening x).

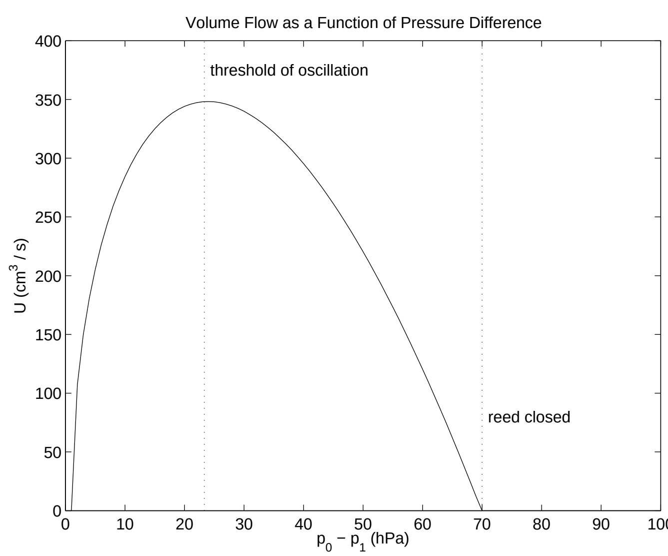


Figure 26: The reed table.