Adding Sinusoids at Different Frequencies

- When adding sinusoids at different frequencies, the resulting signal is no longer sinusoidal.

![Figure 1: Adding 2 sinusoids at different frequencies.](image)

• Is it periodic?

A Non-Sinusoidal Periodic Signal

- When frequencies of added sinusoids are integer multiple of a fundamental frequency $f_0$, signal is periodic with period $T = \frac{1}{f_0}$.

![Image of periodic signal with frequencies 3, 6, 9 Hz](image)

- Adding sinusoids at 3, 6, 9 Hz produces a periodic signal with period $T = \frac{1}{3}$ s.

Spectrum: the Frequency Domain

- When signals are no longer purely sinusoidal, it’s useful to view them in the frequency domain to see their sinusoidal components.

- In the frequency domain, a vertical line indicates the amplitude of frequency components.

![Image of spectrum showing amplitude of frequencies](image)

- Adding unit-amplitude sinusoids at 5, 10, and 15 Hz produces length-one “spikes” at these frequencies.
Harmonics

- Sinusoidal components, or partials, that are integer multiples of a fundamental frequency are called harmonics.

- The even spacing between the “spikes” indicates that higher-frequency sinusoidal components are integer multiples of a lower fundamental frequency.

Harmonic vs. Inharmonic

- Harmonic waveform:

- Inharmonic waveform:

- Signals with harmonic spectra have a fundamental frequency and therefore have a periodic waveform (and vice versa).

Clarinet Analysis

- The clarinet tone (shown here in the steady state) can be viewed in both time and frequency domain.

- Summing sinusoids at 145, 435 and 732 Hz (as well as others), can approximate a synthesis of the clarinet (in the steady state).

Harmonicity

- Harmonicity contributes to our sense of pitch.

  - Tenor sax (middle E)
  - Japanese Bell
  - Snare Drum

Figure 2: Frequency analysis of a clarinet note.

- Summing sinusoids at 145, 435 and 732 Hz (as well as others), can approximate a synthesis of the clarinet (in the steady state).
Inharmonicity

• Generally, inharmonic overtones lack a clear sense of pitch (difficult to hum).
• The perception of pitch may vary with
  – **individuals**: e.g. musical background
  – **context**: sense of pitch with _inharmonic_ tones tends to be clearer when notes are played in succession.
    1. `bellsclip.wav`: bell in isolation—do you hear a pitch?
    2. `bells.wav`: bells in melodic context

Context allows us to focus on the change in notes rather than on any one note itself.

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Standard Periodic Waveforms

- **square wave**:
  – harmonics: $n = 1, 3, 5, \ldots$ (odd)
  – amplitudes: $1/n$
  – phase: $\sin \rightarrow 0$, $\cos \rightarrow -\pi/2$

- **triangle wave**:
  – harmonics: $n = 1, 3, 5, \ldots$ (odd)
  – amplitudes: $1/n^2$
  – phase: $\sin \rightarrow \pi/2$, $\cos \rightarrow 0$

- **sawtooth wave**:
  – harmonics: $n = 1, 2, 3, \ldots$ (all)
  – amplitudes: $1/n$
  – phase: $\sin \rightarrow 0$, $\cos \rightarrow -\pi/2$
Spectra of Standard Waveforms

- Figure 3: Spectra of complex waveforms

Harmonic Series and Just Intonation

- Tones are often compared on the basis of the musical interval separating them: m3, P5, P8 etc.
- Intervals exist within the well-known harmonic series:
  - octave (P8), 2:1 (doubling of frequency)
  - perfect fifth (P5), 3:2
  - perfect fourth (P4), 4:3
  - major third (M3), 5:4
  - etc...
  which is the basis of just intonation.

Equal-tempered tuning

- In equal-tempered tuning, there are 12 evenly spaced tones in an octave, called semi-tones:
  - the frequency $n$ semitones above A440 is $440 \times 2^{n/12}$ Hz.
  - the frequency $n$ semitones below A440 is $440 \times 2^{-n/12}$ Hz.
  - e.g.: a perfect fifth above 440 yields $440 \times 2^{7/12} = 440 \times 1.4983$ Hz.
  ... a slightly different result than when using ratios of harmonic numbers as in just intonation:
  $440 \times 3/2 = 440 \times 1.5$ Hz.

Pitch and Frequency

- Recall, we tend to hear frequency change logarithmically:
- Though an octave between two tones is perceived as being (more or less) the same change in all registers, the difference in Hz increases with increased frequency:
  - P8 above 220 Hz is an increase of 220 Hz.
  - P8 above 440 Hz is an increase of 440 Hz.
Beat Notes

- What happens when we add two frequencies that are close in frequency?

- The waveform shows a periodic, low frequency sinusoidal amplitude envelope on a higher frequency sinusoid creating a beat note.

- This can be explained by the Cosine Product formula:

\[ \cos(a) \cos(b) = \frac{\cos(a + b) + \cos(a - b)}{2} \]

Delivering a Sinusoid

- There is no audible difference in a delayed sinusoid, other than that it sounds later in time.

- But what happens when you sum the original sinusoid and its delayed version?

Interference

- Adding sinusoids can cause a kind of interference:
  - destructive: when the resulting sum shows a reduced amplitude
  - constructive: when the resulting sum shows an increased amplitude

- Consider summing a harmonic signal with a version delayed by half the period of its fundamental frequency:

Cancellation of Harmonics

- With the appropriate delay, there can be complete cancelation of certain harmonics:

- Delay and interference occurs often in acoustic systems:
  - contributes to the presence/absence of partials/harmonics
  - influences the resulting spectrum and perceived timbre of the sound.